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Abstract

We study the transient response of a thermoelastic structure made of two tridimensional bodies connected by a thin adhesive layer. Once more we highlight the powerful flexibility of Trotter's theory of approximation of semi-groups of operators acting on variable spaces: considering the geometrical and physical characteristics of the thin layer as *parameters*, we are able to show in a unitary way that this situation leads to a huge variety of limit models the properties of which are detailed. In particular, according to the relative behaviors of the different parameters involved, new features are evidenced such as the apparition of an added specific heat coefficient for the interface or of additional thermomechanical state variables defined not only on the limit geometric interface but on its cartesian product by any interval of real numbers.

Résumé

On étudie la réponse transitoire d'une structure thermoélastique composée de deux corps tridimensionnels reliés par une fine couche adhésive. À nouveau, la théorie de Trotter d'approximation de semi-groupes d'opérateurs agissant sur des espaces variables montre sa grande flexibilité : en considérant les caractéristiques géométriques et physiques de la couche mince comme des paramètres, on établit de manière unitaire que cette situation conduit à une étonnante variété de modèles limites dont les propriétés sont détaillées. En particulier, en fonction des comportements relatifs des différents paramètres impliqués, des caractéristiques singulières sont mises en évidence, comme l'apparition d'un coefficient de chaleur spécifique ajouté pour l'interface ou de variables d'état thermomécaniques supplémentaires définies non seulement sur l'interface géométrique limite mais aussi sur son produit cartésien par tout intervalle de nombres réels.

Keywords: Bonding problems, linearized thermoelasticity, transient problems, m-dissipative operators, asymptotic mathematical modeling, approximation of semi-groups in the sense of Trotter.

1. Setting the problem

We pursue our investigations on thin junctions initiated in [1, 2], then further developped in [3–11], and hereafter consider the situation of a transient *multi-physical coupling* within the scope of linear thermoelasticity.

Let $\{ e_1, e_2, e_3 \}$ be an orthonormal basis of $\mathbb{R}^3$ assimilated to the physical Euclidean space. For all $\xi = (\xi_1, \xi_2, \xi_3)$ in $\mathbb{R}^3$, $\widehat{\xi}$ stands for (ξ_1, ξ_2). The space of all $(n \times n)$ symmetric matrices is denoted by $\mathbb{S}^n$ and equipped with the usual inner product and norm denoted by $\cdot$ and $| \ |$ (as in $\mathbb{R}^3$). The space of linear symmetric mappings from $\mathbb{S}^n$ into $\mathbb{S}^n$ is denoted by $\mathrm{Lin}(\mathbb{S}^n)$. For all η in $\mathbb{S}^3$, $\widehat{\eta}$ stands for the matrix $(\eta_{\alpha\beta})_{1 \le \alpha, \beta \le 2}$ in $\mathbb{S}^2$. We study the dynamic

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response of a linearly thermoelastic structure consisting of two adhering bodies connected by a thin adhesive layer and subjected to a given loading. Let Ω be a domain of $\mathbb{R}^3$ with Lipschitz-continuous boundary $\partial\Omega$. The intersection of Ω with $\{x_3 = 0\}$ is a domain S of $\mathbb{R}^2$ with positive two-dimensional Hausdorff measure $\mathcal{H}_2(S)$. Let ε be a positive number and $\Omega^\pm := \Omega \cap \{\pm x_3 > 0\}$, then the adhesive and the adhering bodies occupy $B_\varepsilon := S \times (-\varepsilon, +\varepsilon)$ and $\Omega^\pm_\varepsilon := \Omega^\pm \pm \varepsilon e_3$, respectively; we define $\Omega_\varepsilon := \Omega^+_\varepsilon \cup \Omega^-_\varepsilon$, $S^\pm_\varepsilon := S \pm \varepsilon e_3$ and $O_\varepsilon := \Omega_\varepsilon \cup B_\varepsilon \cup S^+_\varepsilon \cup S^-_\varepsilon$. We consider two partitions $(\Gamma^{\mathcal{M}\mathrm{D}}, \Gamma^{\mathcal{M}\mathrm{N}})$, $(\Gamma^{\mathcal{T}\mathrm{D}}, \Gamma^{\mathcal{T}\mathrm{N}})$ of $\partial\Omega$, and for all elements Γ of these two partitions, the sets $\Gamma^\pm$, $\Gamma^\pm_\varepsilon$ and Γ_ε respectively denote $\Gamma \cap \{\pm x_3 > 0\}$, $\Gamma^\pm \pm \varepsilon e_3$ and $\Gamma^+_\varepsilon \cup \Gamma^-_\varepsilon$. Moreover, we assume that $\mathcal{H}_2(\Gamma^{\mathcal{M}\mathrm{D}+})$ and $\mathcal{H}_2(\Gamma^{\mathcal{T}\mathrm{D}+})$ are positive. The contact between the adhesive and the two adhering bodies is assumed to be perfect from both thermal and mechanical points of view. The structure is clamped on $\Gamma^{\mathcal{M}\mathrm{D}}_\varepsilon$, subjected to body forces of density f_ε and surface forces of density $g^{\mathcal{M}}_\varepsilon$ on $\Gamma^{\mathcal{M}\mathrm{N}}_\varepsilon$, it is maintained at a uniform temperature T_0 on $\Gamma^{\mathcal{T}\mathrm{D}}_\varepsilon \cup \gamma^{\mathcal{T}\mathrm{D}} \times (-\varepsilon, \varepsilon)$ and subjected to a thermal flux $g^{\mathcal{T}}_\varepsilon$ on $\Gamma^{\mathcal{T}\mathrm{N}}_\varepsilon \cup \gamma^{\mathcal{T}\mathrm{N}} \times (-\varepsilon, \varepsilon)$, where $(\gamma^{\mathcal{T}\mathrm{D}}, \gamma^{\mathcal{T}\mathrm{N}})$ is a partition of ∂S. The whole structure is modeled as linearly thermoelastic in the following way. Let $(\rho_L, \mu_L, \beta_L, \kappa_L, \alpha_L)$ in $(0, +\infty)^5$, a_L in $\mathrm{Lin}(\mathbb{S}^3)$, $d := (\rho, \beta, \alpha, \kappa, a)$ in $L^\infty(\Omega; \mathbb{R} \times \mathbb{R} \times \mathbb{S}^3 \times \mathbb{S}^3 \times \mathrm{Lin}(\mathbb{S}^3))$ satisfying

$$\begin{cases} \alpha(x) \geq 0 \text{ a.e. } x \in \Omega, \\ \exists c > 0 \text{ s.t. } \rho(x), \beta(x) \geq c, \quad \kappa(x)\xi \cdot \xi \geq c|\xi|^2 \ \forall \xi \in \mathbb{R}^3, \quad a_L e \cdot e,\ a(x)e \cdot e \geq c|e|^2 \ \forall e \in \mathbb{S}^3, \text{ a.e } x \in \Omega. \end{cases} \tag{1.1}$$

The symbols $\rho_L, \beta_L, \alpha_L, \kappa_L, \mu_L a_L$ respectively represent the mass density, the specific heat coefficient, the thermal dilatation, the thermal conductivity, and the elasticity tensor of the adhesive, while $d_\varepsilon = (\rho_\varepsilon, \beta_\varepsilon, \alpha_\varepsilon, \kappa_\varepsilon, a_\varepsilon)$ denotes the analogous quantities for the adhering bodies with

$$d_\varepsilon(x) := d(x \mp \varepsilon e_3) \quad \text{a.e. } x \in \Omega^\pm_\varepsilon. \tag{1.2}$$

Similarly

$$\begin{cases} \exists (f, g^{\mathcal{M}}, g^{\mathcal{T}}) \in L^2(\Omega; \mathbb{R}^3) \times L^2(\Gamma^{\mathcal{M}\mathrm{N}}; \mathbb{R}^3) \times L^2(\Gamma^{\mathcal{T}\mathrm{N}}) \text{ s.t.} \\ f_\varepsilon(x) = f(x \mp \varepsilon e_3) \quad \text{a.e. } x \in \Omega^\pm_\varepsilon, \quad f_\varepsilon(x) = 0 \quad \text{a.e. } x \in B_\varepsilon, \\ g^{\mathcal{M}}_\varepsilon(x) = g^{\mathcal{M}}(x \mp \varepsilon e_3) \quad \text{a.e. } x \in \Gamma^{\mathcal{M}\mathrm{N}}_\varepsilon, \quad g^{\mathcal{M}}_\varepsilon(x) = 0 \quad \text{a.e. } x \in \partial S \times (-\varepsilon, \varepsilon), \\ g^{\mathcal{T}}_\varepsilon(x) = g^{\mathcal{T}}(x \mp \varepsilon e_3) \quad \text{a.e. } x \in \Gamma^{\mathcal{T}\mathrm{N}}_\varepsilon, \quad g^{\mathcal{T}}_\varepsilon(x) = 0 \quad \text{a.e. } x \in \gamma^{\mathcal{T}\mathrm{N}} \times (-\varepsilon, \varepsilon). \end{cases} \tag{1.3}$$

Thus the problem $(\mathcal{P}_s)$ of determining the evolution in the framework of small perturbations of the assembly, whose state is denoted by $U_s = (u_s, v_s, \theta_s)$, u_s, v_s, θ_s being the fields of displacement, velocity and temperature increment with respect to T_0, involves a sextuplet $s := (\varepsilon, \rho_L, \mu_L, \beta_L, \kappa_L, \alpha_L)$ of data so that all the fields will be thereafter indexed by s. If $U^0_s = (u^0_s, v^0_s, \theta^0_s)$ is the given initial state, a formulation of $(\mathcal{P}_s)$ could be:

$$(\mathcal{P}_s) \quad \begin{cases} \text{Find } U_s \text{ sufficiently smooth in } O_\varepsilon \times [0, T] \text{ such that } u_s = 0 \text{ on } \Gamma^{\mathcal{M}\mathrm{D}}_\varepsilon,\ \theta_s = 0 \text{ on } \Gamma^{\mathcal{T}\mathrm{D}}_\varepsilon \cup \gamma^{\mathcal{T}\mathrm{D}} \times (-\varepsilon, \varepsilon) \\ U_s(0) = U^0_s \text{ satisfying:} \\ \displaystyle\int_{\Omega_\varepsilon} \rho_\varepsilon \frac{\partial v_s}{\partial t} \cdot v' + a_\varepsilon(e(u_s) - \theta_s \alpha_\varepsilon) \cdot e(v')\, dx + \int_{B_\varepsilon} \rho_L \frac{\partial v_s}{\partial t} \cdot v' + \mu_L a_L(e(u_s) - \alpha_L \theta_s I) \cdot e(v')\, dx \\ \displaystyle = \int_{\Omega_\varepsilon} f_\varepsilon \cdot v'\, dx + \int_{\Gamma^{\mathcal{M}\mathrm{N}}_\varepsilon} g^{\mathcal{M}}_\varepsilon \cdot v'\, d\mathcal{H}_2, \\ \displaystyle\int_{\Omega_\varepsilon} \beta_\varepsilon \frac{\partial \theta_s}{\partial t} \theta' + \kappa_\varepsilon \nabla\theta_s \cdot \nabla\theta' + (a_\varepsilon \alpha_\varepsilon \cdot e(v_s))\theta'\, dx + \int_{B_\varepsilon} \beta_L \frac{\partial \theta_s}{\partial t} \theta' + \kappa_L \nabla\theta_s \cdot \nabla\theta' + \mu_L \alpha_L (a_L I \cdot e(v_s))\theta'\, dx \\ \displaystyle = \int_{\Gamma^{\mathcal{T}\mathrm{N}}_\varepsilon} g^{\mathcal{T}}_\varepsilon \theta'\, d\mathcal{H}_2 \\ \text{for all } (v', \theta') \text{ sufficiently smooth in } O_\varepsilon \text{ and vanishing on } \Gamma^{\mathcal{M}\mathrm{D}}_\varepsilon \times \left(\Gamma^{\mathcal{T}\mathrm{D}}_\varepsilon \cup \gamma^{\mathcal{T}\mathrm{D}} \times (-\varepsilon, \varepsilon)\right), \end{cases}$$

where t denotes the time, $e(u)$ is the linearized strain associated with the field of displacement u, and I is the identity matrix of $\mathbb{S}^3$.

2. Existence and uniqueness of a solution to ($\mathcal{P}_s$)

Assuming

$$(f, g^{\mathcal{M}}, g^{\mathcal{T}}) \in C^{0,1}\big([0,T]; L^2(\Omega;\mathbb{R}^3)\big) \times C^{1,1}\big([0,T]; L^2(\Gamma^{\mathcal{M}\mathrm{N}};\mathbb{R}^3)\big) \times C^{1,1}\big([0,T]; L^2(\Gamma^{\mathcal{T}\mathrm{N}})\big), \tag{H1}$$

we seek $z_s = (u_s, \theta_s)$ in the form

$$z_s = z_s^e + z_s^r, \tag{2.1}$$

where z_s^e is the unique solution to

$$z_s^e(t) \in Z_s; \quad \Phi_s(z_s^e(t), z') = L_\varepsilon(t)(z') \quad \forall z' \in Z_s,\ \forall t \in [0,T], \tag{2.2}$$

with

$$Z_s := H^1_{\Gamma_\varepsilon^{\mathcal{M}\mathrm{D}}}(O_\varepsilon;\mathbb{R}^3) \times H^1_{\Gamma_\varepsilon^{\mathcal{T}\mathrm{D}}}(O_\varepsilon), \tag{2.3}$$

where for all open set G of $\mathbb{R}^N$, $N = 1$ or 3, $H^1_\gamma(G;\mathbb{R}^N)$ denotes the subset of the Sobolev space $H^1(G;\mathbb{R}^N)$ of elements with vanishing trace on γ included in ∂G,

$$\Phi_s(z, z') := (u, u')_{1,s} + (\theta, \theta')_{4,s} - (u', \theta)_{5,s} + (u, \theta')_{5,s} \quad \forall z = (u, \theta),\ \forall z' = (u', \theta') \in Z_s, \tag{2.4}$$

$$\begin{cases} (u, u')_{1,s} := \displaystyle\int_{\Omega_\varepsilon} a_\varepsilon e(u) \cdot e(u')\, dx + \int_{B_\varepsilon} \mu_L a_L e(u) \cdot e(u')\, dx & \forall u, u' \in H^1_{\Gamma_\varepsilon^{\mathcal{M}\mathrm{D}}}(O_\varepsilon;\mathbb{R}^3), \\ (\theta, \theta')_{4,s} := \displaystyle\int_{\Omega_\varepsilon} \kappa_\varepsilon \nabla\theta \cdot \nabla\theta'\, dx + \int_{B_\varepsilon} \kappa_L \nabla\theta \cdot \nabla\theta'\, dx & \forall \theta, \theta' \in H^1_{\Gamma_\varepsilon^{\mathcal{T}\mathrm{D}}}(O_\varepsilon), \\ (u, \theta)_{5,s} := \displaystyle\int_{\Omega_\varepsilon} (a_\varepsilon \alpha_\varepsilon \cdot e(u))\theta\, dx + \int_{B_\varepsilon} \mu_L \alpha_L (a_L I \cdot e(u))\theta\, dx & \forall (u, \theta) \in H^1_{\Gamma_\varepsilon^{\mathcal{M}\mathrm{D}}}(O_\varepsilon;\mathbb{R}^3) \times H^1_{\Gamma_\varepsilon^{\mathcal{T}\mathrm{D}}}(O_\varepsilon), \end{cases} \tag{2.5}$$

and

$$L_\varepsilon(t)(z') := \int_{\Gamma_\varepsilon^{\mathcal{M}\mathrm{N}}} g_\varepsilon^{\mathcal{M}} \cdot u'\, d\mathcal{H}_2 + \int_{\Gamma_\varepsilon^{\mathcal{T}\mathrm{N}}} g_\varepsilon^{\mathcal{T}} \theta'\, dx \quad \forall z' \in Z_s,\ \forall t \in [0,T], \tag{2.6}$$

As $(g^{\mathcal{M}}, g^{\mathcal{T}}) \mapsto z_s^e$ is linear continuous from $L^2(\Gamma^{\mathcal{M}\mathrm{N}};\mathbb{R}^3) \times L^2(\Gamma^{\mathcal{T}\mathrm{N}})$ into Z_s, we have

$$z_s^e \in C^{1,1}([0,T]; Z_s). \tag{2.7}$$

The remaining part z_s^r of z_s will be involved in an evolution equation governed by an m-dissipative operator A_s in a Hilbert space $\mathfrak{H}_s$ of possible states with finite thermomechanical (strain, kinetic, thermal) energy defined by

$$\mathfrak{H}_s := H^1_{\Gamma_\varepsilon^{\mathcal{M}\mathrm{D}}}(O_\varepsilon;\mathbb{R}^3) \times L^2(O_\varepsilon;\mathbb{R}^3) \times L^2(O_\varepsilon), \tag{2.8}$$

and endowed with the following inner product and norm:

$$(U, U')_s := (u, u')_{1,s} + K_s((v, \theta), (v', \theta')), \quad |U|_s^2 := (U, U)_s, \tag{2.9}$$

$$K_s((v, \theta), (v', \theta')) = (v, v')_{2,s} + (\theta, \theta')_{3,s} \quad \forall U = (u, v, \theta),\ \forall U' = (u', v', \theta') \in \mathfrak{H}_s, \quad \text{with} \tag{2.10}$$

$$\begin{cases} (v, v')_{2,s} := \displaystyle\int_{\Omega_\varepsilon} \rho_\varepsilon v \cdot v'\, dx + \int_{B_\varepsilon} \rho_L v \cdot v'\, dx, \\ (\theta, \theta')_{3,s} := \displaystyle\int_{\Omega_\varepsilon} \beta_\varepsilon \theta\theta'\, dx + \int_{B_\varepsilon} \beta_L \theta\theta'\, dx. \end{cases} \tag{2.11}$$

Operator A_s is defined by

$$\begin{cases} D(A_s) := \left\{ U = (u, v, \theta) \in \mathfrak{H}_s; \left\{ \begin{array}{ll} \text{i)} & (v, \theta) \in Z_s \text{ and} \\ \text{ii)} & \exists!\ (w, \tau) \in L^2(O_\varepsilon; \mathbb{R}^3) \times L^2(O_\varepsilon) \text{ s.t.} \\ & \quad (w, v')_{2,s} + (u, v')_{1,s} - (v', \theta)_{5,s} = 0 \quad \forall v' \in H^1_{\Gamma_\varepsilon^{\mathcal{M}\mathrm{D}}}(O_\varepsilon; \mathbb{R}^3), \\ & \quad (\tau, \theta')_{3,s} + (\theta, \theta')_{4,s} + (v, \theta')_{5,s} = 0 \quad \forall \theta' \in H^1_{\Gamma_\varepsilon^{\mathcal{T}\mathrm{D}}}(O_\varepsilon), \end{array} \right. \right\} \\ A_s U = (v, w, \tau). \end{cases} \tag{2.12}$$

It is straightforward to check the following.

Proposition 2.1. *Operator A_s is m-dissipative, and for all $\phi_s = (\phi_s^1, \phi_s^2, \phi_s^3)$ in $\mathfrak{H}_s$,*

$$\begin{cases} \overline{U}_s = (\overline{u}_s, \overline{v}_s, \overline{\theta}_s) \text{ s.t.} \\ \overline{U}_s - A_s \overline{U}_s = \phi_s \end{cases} \iff \begin{cases} \overline{u}_s = \overline{v}_s + \phi_s^1 \\ \overline{z}_s = (\overline{v}_s, \overline{\theta}_s) \in Z_s; \quad \Psi_s(\overline{z}_s, z) = \mathcal{L}_s(z) \quad \forall z = (v, \theta) \in Z_s \\ \textit{with} \\ \Psi_s = \Phi_s + K_s \\ \mathcal{L}_s(z) = -(\phi_s^1, u)_{1,s} + K_s((\phi_s^2, \phi_s^3), z) \quad \forall z = (v, \theta) \in Z_s. \end{cases} \tag{2.13}$$

Then, taking into account (H1), (2.1), (2.2), (2.7), (2.12), it is clear that $(\mathcal{P}_s)$ is "formally equivalent" to

$$\begin{cases} \dfrac{dU_s^r}{dt} - A_s U_s^r = F_s := \left(u_s^e - \dfrac{du_s^e}{dt}, -\dfrac{du_s^e}{dt} + f_s, -\dfrac{d\theta_s^e}{dt} \right), \\ U_s^r(0) = U_s^0 - \left(u_s^e(0), 0, \theta_s^e(0) \right), \end{cases} \tag{2.14}$$

with f_s equal to $f_\varepsilon / \rho_\varepsilon$ in Ω_ε and 0 in B_ε. Consequently ([12]) the following holds:

Theorem 2.1. *If $(f, g^{\mathcal{M}}, g^{\mathcal{T}})$ satisfies* (H1) *and U_s^0 belongs to $(u_s^e(0), 0, \theta_s^e(0)) + D(A_s)$, then* (2.14) *has a unique solution such that U_s^r belongs to $C^1([0, T]; \mathfrak{H}_s)$. Hence there exists a unique (u_s, θ_s) in*

$$\left(C^1\left([0, T]; H^1_{\Gamma_\varepsilon^{\mathcal{M}\mathrm{D}}}(O_\varepsilon; \mathbb{R}^3)\right) \cap C^2\left([0, T]; L^2(O_\varepsilon; \mathbb{R}^3)\right) \right) \times \left(C^1\left([0, T]; L^2(O_\varepsilon)\right) \cap C^0\left([0, T]; H^1_{\Gamma_\varepsilon^{\mathcal{T}\mathrm{D}}}(O_\varepsilon)\right) \right),$$

which does satisfy

$$\begin{aligned} &\left(\frac{d^2 u_s}{dt^2}, u \right)_{2,s} + (u_s, u)_{1,s} - (u, \theta_s)_{5,s} = \int_{\Omega_\varepsilon} f_\varepsilon \cdot u \, dx + \int_{\Gamma_\varepsilon^{\mathcal{M}\mathrm{N}}} g_\varepsilon^{\mathcal{M}} \cdot u \, d\mathcal{H}_2 \quad \forall u \in H^1_{\Gamma_\varepsilon^{\mathcal{M}\mathrm{D}}}(O_\varepsilon; \mathbb{R}^3), \\ &\left(\frac{d\theta_s}{dt}, \theta \right)_{3,s} + (\theta_s, \theta)_{4,s} + \left(\frac{du_s}{dt}, \theta \right)_{5,s} = \int_{\Gamma_\varepsilon^{\mathcal{T}\mathrm{N}}} g_\varepsilon^{\mathcal{T}} \theta \, d\mathcal{H}_2 \quad \forall \theta \in H^1_{\Gamma_\varepsilon^{\mathcal{T}\mathrm{D}}}(O_\varepsilon). \end{aligned} \tag{2.15}$$

We set

$$U_s^e = (u_s^e, z_s^e). \tag{2.16}$$

3. A mathematical analysis of the asymptotic behavior

Now we regard the sextuplet s of geometrical and thermomechanical *data* as a sextuplet of *parameters* taking values in a countable subset of $(0, +\infty)^6$ with a single cluster point $\overline{s}$ in $\{0\} \times [0, +\infty]^5$ and study the asymptotic behavior of U_s in order to suggest a simplified but accurate enough model for the genuine physical situation. We will show that, depending on the relative behavior of $(\rho_L, \mu_L, \beta_L, \kappa_L, \alpha_L)$ with respect to ε, numerous (100!) limit models appear. They are indexed by $\mathrm{I} = (\mathrm{I_M}, \mathrm{I_T}) \in (\{0, 1\} \times \{0, 1, 2, 3, 4\})^2$, $\mathrm{I_M} = (\mathrm{I_{M1}}, \mathrm{I_{M2}})$, $\mathrm{I_T} = (\mathrm{I_{T1}}, \mathrm{I_{T2}})$ defined as follows.

Let $r^{\rm M}_0 = r^{\rm M}_1 = r^{\rm M}_2 = 1$, $r^{\rm M}_3 = -1$, $r^{\rm M}_4 = -3$, $r^{\rm T}_0 = r^{\rm T}_1 = r^{\rm T}_2 = 1$, $r^{\rm T}_3 = r^{\rm T}_4 = -1$, we assume:

$$\begin{cases} \text{There exists } (\bar\rho_L, \bar\beta_L, \bar\mu^{\rm I}_L, \bar\kappa^{\rm I}_L) \text{ in } [0,+\infty)^2 \times [0,+\infty]^2 \text{ such that:} \\ (\bar\rho_L, \bar\beta_L) := \lim_{s\to\bar s} \varepsilon\, (\rho_L, \beta_L)\,, \\ (\bar\mu^{\rm I}_L, \bar\kappa^{\rm I}_L) := \lim_{s\to\bar s} \left(\mu_L/\varepsilon^{r^{\rm M}_{\rm I_{M2}}}, \kappa_L/\varepsilon^{r^{\rm T}_{\rm I_{T2}}}\right) \text{ with } \overline{\lim_{s\to\bar s}}\, \varepsilon^2 \left(\mu_L^{-1}, \kappa_L^{-1}\right) \in [0,+\infty)^2, \\ \text{and} \\ {\rm I_{M1}}, {\rm I_{T1}} = 0 \text{ if } \bar\rho_L, \bar\beta_L = 0, \quad {\rm I_{M1}}, {\rm I_{T1}} = 1 \text{ if } \bar\rho_L > 0,\ \bar\beta_L > 0, \\ \bar\mu^{\rm I}_L = 0 \text{ and } \mathcal{H}_2(\Gamma^{\mathcal{M}{\rm D}-}) > 0 \text{ when } {\rm I_{M2}} = 0, \quad \bar\mu^{\rm I}_L \in (0,+\infty) \text{ when } {\rm I_{M2}} = 1, \\ \bar\mu^{\rm I}_L = +\infty \text{ and } \lim_{s\to\bar s} \varepsilon\mu_L = 0 \text{ when } {\rm I_{M2}} = 2, \quad \bar\mu^{\rm I}_L \in (0,+\infty) \text{ when } {\rm I_{M2}} = 3,4, \\ \bar\kappa^{\rm I}_L = 0 \text{ and } \mathcal{H}_2(\Gamma^{\mathcal{T}{\rm D}-}) > 0 \text{ when } {\rm I_{T2}} = 0, \quad \bar\kappa^{\rm I}_L \in (0,+\infty) \text{ when } {\rm I_{T2}} = 1, \\ \bar\kappa^{\rm I}_L = +\infty \text{ and } \lim_{s\to\bar s} \varepsilon\kappa_L = 0 \text{ when } {\rm I_{T2}} = 2, \quad \bar\kappa^{\rm I}_L \in (0,+\infty) \text{ when } {\rm I_{T2}} = 3, \\ \bar\kappa^{\rm I}_L = +\infty \text{ when } {\rm I_{T2}} = 4. \end{cases} \tag{H2}$$

3.1. A candidate for the limit behavior

From now on, C denotes various constants which may vary from line to line and we use the convention $0 \times \infty = \infty \times 0 = 0$.

3.1.1. The limit space $\mathfrak{H}^{\rm I}$

This candidate could be determined by studying the asymptotic behavior of sequences with bounded total thermomechanical energy. It will appear that in some cases the thermomechanical state of the "limit structure", where the three-dimensional adhesive layer is *geometrically* replaced by the surface S, it shrinks to, does not involve the sole state variables of the adhering bodies but additional thermomechanical state variables not necessarily defined on S *but in* $B := S \times (-1,1)$ which accounts for the limit behavior of the adhesive layer.

It is convenient to introduce the following "scaling operators" which transform a field y_s defined on B_ε into a field y_{sB} defined on B in such a way that a bounded energy for y_s is equivalent to a bounded "scaled" energy for y_{sB}:

Operators $\mathcal{S}^{u,{\rm I_{M2}}}_\varepsilon$:
$u \in H^1(B_\varepsilon;\mathbb{R}^3) \mapsto u_B = \mathcal{S}^{u,{\rm I_{M2}}}_\varepsilon u \in H^1(B;\mathbb{R}^3)$ defined by

$$\begin{cases} {\rm I_{M2}} = 0,1,2 & u_B(x) = u(\widehat{x}, \varepsilon x_3) \quad \text{a.e. } x \in B, \\ & \text{and } e^{\rm I_{M2}}_{ij}(\varepsilon, u_B) := \begin{cases} \varepsilon e_{ij}(u_B), & 1 \le i,j \le 2 \\ (\varepsilon\partial_i u_{B3} + \partial_3 u_{Bi})/2, & i = 1,2,\ j = 3 \\ \partial_{33} u_{B3}, & i = j = 3, \end{cases} \\ {\rm I_{M2}} = 3 & \widehat{u}_B(x) = \widehat{u}(\widehat{x}, \varepsilon x_3), \quad u_{B3}(x) = u_3(\widehat{x}, \varepsilon x_3)/\varepsilon \quad \text{a.e. } x \in B, \\ & \text{and } e^3_{ij}(\varepsilon, u_B) := \begin{cases} e_{ij}(u_B), & 1 \le i,j \le 2 \\ (1/\varepsilon) e_{i3}(u_B), & 1 \le i \le 2,\ j = 3 \\ (1/\varepsilon^2) e_{33}(u_B), & i = j = 3, \end{cases} \\ {\rm I_{M2}} = 4 & \widehat{u}_B(x) = \widehat{u}(\widehat{x}, \varepsilon x_3)/\varepsilon, \quad u_{B3}(x) = u_3(\widehat{x}, \varepsilon x_3) \quad \text{a.e. } x \in B, \\ & \text{and } e^4_{ij}(\varepsilon, u_B) = e^3_{ij}(\varepsilon, u_B). \end{cases} \tag{3.1}$$

Clearly:

$$\mu_L \int_{B_\varepsilon} |e(u)|^2\, dx = \left(\mu_L / r^{\rm M}_{\rm I_{M2}}\right) \int_B \left| e^{\rm I_{M2}}(\varepsilon, \mathcal{S}^{u,{\rm I_{M2}}}_\varepsilon u)\right|^2 dx. \tag{3.2}$$

Operators $\mathcal{S}_\varepsilon^v, \mathcal{S}_\varepsilon^\theta$:
$(v,\theta) \in L^2(B_\varepsilon;\mathbb{R}^3\times\mathbb{R}) \mapsto (v_B,\theta_B) = (\mathcal{S}_\varepsilon^v v, \mathcal{S}_\varepsilon^\theta \theta) \in L^2(B;\mathbb{R}^3\times\mathbb{R})$ defined by

$$(v_B(x),\theta_B(x)) = (v(\widehat{x},\varepsilon x_3),\theta(\widehat{x},\varepsilon x_3)) \quad \text{a.e. } x\in B, \tag{3.3}$$

which satisfy

$$\left(\rho_L\int_{B_\varepsilon}|v(x)|^2\,dx,\ \beta_L\int_{B_\varepsilon}|\theta(x)|^2\,dx\right) = \left(\rho_L\varepsilon\int_B\left|\mathcal{S}_\varepsilon^v v\right|^2\,dx,\ \beta_L\varepsilon\int_B\left|\mathcal{S}_\varepsilon^\theta\theta\right|^2\,dx\right). \tag{3.4}$$

In view of following Proposition 3.1, it is natural to recall some classical notions. Let

$$\Omega^i := \Omega\setminus S \quad \text{if } i=0,1, \quad \Omega^i := \Omega \quad \text{if } i=2,3,4. \tag{3.5}$$

For an element u of $H^1(\Omega\setminus S;\mathbb{R}^N)$, $N=1$ or 3, we will denote its restrictions to the open sets $\Omega^\pm$ by $u^\pm$ which is an element of $H^1(\Omega^\pm;\mathbb{R}^N)$. The symbols $\gamma_S(u^+)$ and $\gamma_S(u^-)$ will denote the trace of u^+ and u^-, respectively, on the set S. Of course, for u in $H^1(\Omega;\mathbb{R}^N)$, $\gamma_S(u)$ will denote the trace of u on S. We also use

$$[\![u]\!] = \gamma_S(u^+) - \gamma_S(u^-), \tag{3.6}$$

$$H_{\partial_3}(B;\mathbb{R}^N) := \left\{ u\in L^2(B;\mathbb{R}^N) \text{ s.t. } \partial_3 u\in L^2(B;\mathbb{R}^N)\right\} \quad N=1 \text{ or } 3, \tag{3.7}$$

it is well-known that a continuous mapping $\gamma_{S^\pm}$ is defined on $H_{\partial_3}(B;\mathbb{R}^N)$ for the traces on $S^\pm := S\pm e_3$ with values in $L^2(S^\pm;\mathbb{R}^N)$, and, from now on, $\gamma_{S^\pm}(u)$ is treated as an element of $L^2(S;\mathbb{R}^N)$,

$$\begin{aligned} V_{\mathrm{KL}}(B) := \big\{ & u\in H^1(B;\mathbb{R}^3);\ \exists(u^M,u^F)\in H^1(S;\mathbb{R}^2)\times H^2(S) \text{ s.t.} \\ & \widehat{u}(x) = u^M(\widehat{x}) - x_3\widehat{\nabla}u^F(\widehat{x}), \quad u_3(x) = u^F(\widehat{x}) \text{ a.e. } x\in B \big\} \\ = \{ & u\in H^1(B;\mathbb{R}^3);\ e_{i3}=0 \text{ a.e. in } B,\ 1\le i\le 3\}. \end{aligned} \tag{3.8}$$

We will use the following Hilbert spaces and norms:

space of displacement fields $\mathfrak{H}_u^{\mathrm{I}}$:

- $\mathrm{I}_{\mathrm{M2}}=0$ $\quad \mathfrak{H}_u^{\mathrm{I}} := \left\{ u^{\mathrm{I}} = (u_\Omega, e_B^u)\in H^1_{\Gamma_{\mathcal{M}\mathrm{D}}}(\Omega^0;\mathbb{R}^3)\times\{0\}\right\}$,
- $\mathrm{I}_{\mathrm{M2}}=1$ $\quad \mathfrak{H}_u^{\mathrm{I}} := \left\{ u^{\mathrm{I}} = (u_\Omega, e_B^u)\in H^1_{\Gamma_{\mathcal{M}\mathrm{D}}}(\Omega^1;\mathbb{R}^3)\times L^2(B;\mathbb{S}^3) \text{ s.t.}\right.$
  $\left.\exists u_B\in H_{\partial_3}(B;\mathbb{R}^3);\ e_B^u = \partial_3 u_B\otimes_S e_3, \gamma_{S^\pm}(u_B) = \gamma_S(u_\Omega^\pm)\right\}$,
- $\mathrm{I}_{\mathrm{M2}}=2$ $\quad \mathfrak{H}_u^{\mathrm{I}} := \left\{ u^{\mathrm{I}} = (u_\Omega, e_B^u)\in H^1_{\Gamma_{\mathcal{M}\mathrm{D}}}(\Omega;\mathbb{R}^3)\times L^2(B;\mathbb{S}^3) \text{ s.t.}\right.$
  $\left.\exists u_B\in H^1(B;\mathbb{R}^3); 0 = e_B^u = \partial_3 u_B\otimes_S e_3, \gamma_S^\pm(u_B) = \gamma_S(u_\Omega)\right\}$,
- $\mathrm{I}_{\mathrm{M2}}=3$ $\quad \mathfrak{H}_u^{\mathrm{I}} := \left\{ u^{\mathrm{I}} = (u_\Omega, e_B^u)\in H^1_{\Gamma_{\mathcal{M}\mathrm{D}}}(\Omega;\mathbb{R}^3)\times L^2(B;\mathbb{S}^3) \text{ s.t.}\right.$
  $\left.\gamma_S(u_{\Omega 3}) = C, \exists u_B\in V_{\mathrm{KL}}(B);\ u_B^M = \gamma_S(\widehat{u}_\Omega), \widehat{\nabla}u_B^F = 0, \widehat{e_B^u} := \widehat{e(u_B)}\right\}$,
- $\mathrm{I}_{\mathrm{M2}}=4$ $\quad \mathfrak{H}_u^{\mathrm{I}} := \left\{ u^{\mathrm{I}} = (u_\Omega, e_B^u)\in H^1_{\Gamma_{\mathcal{M}\mathrm{D}}}(\Omega;\mathbb{R}^3)\times L^2(B;\mathbb{S}^3) \text{ s.t.}\right.$
  $\left.\widehat{e}(\gamma_S(\widehat{u}_\Omega)) = 0,\ \exists u_B\in V_{\mathrm{KL}}(B);\ u_B^F = \gamma_S(u_{\Omega 3}),\ \widehat{e_B^u} := \widehat{e(u_B)}\right\}$,

and they clearly are complete for the norms

$$(u^{\mathrm{I}},u^{\mathrm{I}})_1^{\mathrm{I}} := \int_{\Omega^{\mathrm{I}_{\mathrm{M2}}}} ae(u_\Omega)\cdot e(u_\Omega)\,dx + \bar{\mu}_L^{\mathrm{I}}\int_B a_L e_B^u\cdot e_B^u\,dx \quad \forall \mathrm{I}_{\mathrm{M2}} = 0,1,\ldots,4. \tag{3.9}$$

Note that for a field w in $H^1(S;\mathbb{R}^2)$ we also denote the symmetrized gradient of w by $\widehat{e}(w)$.

space of velocity fields $\mathfrak{H}_v^{\mathrm{I}}$:

- $\mathrm{I_{M1}} = 0 \quad \mathfrak{H}^{\mathrm{I}}_v := \left\{ v^{\mathrm{I}} = v_\Omega \in L^2(\Omega;\mathbb{R}^3) \right\}, \quad (v^{\mathrm{I}}, v^{\mathrm{I}})^{\mathrm{I}}_2 := \int_\Omega \rho v_\Omega \cdot v_\Omega \, dx,$
- $\mathrm{I_{M1}} = 1 \quad \mathfrak{H}^{\mathrm{I}}_v := \left\{ v^{\mathrm{I}} = (v_\Omega, v_B) \in L^2(\Omega;\mathbb{R}^3) \times L^2(B;\mathbb{R}^3) \right\}, \quad (v^{\mathrm{I}}, v^{\mathrm{I}})^{\mathrm{I}}_2 := \int_\Omega \rho v_\Omega \cdot v_\Omega \, dx + \bar\rho_L \int_B |v_B|^2 \, dx,$

space of temperature fields $\mathfrak{H}^{\mathrm{I}}_\theta$:

- $\mathrm{I_{T1}} = 0 \quad \mathfrak{H}^{\mathrm{I}}_\theta := \left\{ \theta^{\mathrm{I}} = \theta_\Omega \in L^2(\Omega) \right\}, \quad (\theta^{\mathrm{I}}, \theta^{\mathrm{I}})^{\mathrm{I}}_3 := \int_\Omega \beta \theta_\Omega^2 \, dx,$
- $\mathrm{I_{T1}} = 1 \quad \mathfrak{H}^{\mathrm{I}}_\theta := \left\{ \theta^{\mathrm{I}} = (\theta_\Omega, \theta_B) \in L^2(\Omega) \times L^2(B) \right\}, \quad (\theta^{\mathrm{I}}, \theta^{\mathrm{I}})^{\mathrm{I}}_3 := \int_\Omega \beta \theta_\Omega^2 \, dx + \bar\beta_L \int_B \theta_B^2 \, dx.$

So, if T_ε is the operator from $L^2(\Omega;\mathbb{R}^N)$ into $L^2(\Omega;\mathbb{R}^N)$, $N = 1$ or 3, defined by

$$(T_\varepsilon w)(x) := w(x \pm \varepsilon e_3), \quad \forall x \in \Omega^\pm, \tag{3.10}$$

we have

Proposition 3.1. *For all sequences $U_s = (u_s, v_s, \theta_s)$ in $\mathfrak{H}_s$ such that $|U_s|_s$ is bounded, there exists $U^{\mathrm{I}} = (u^{\mathrm{I}}, \theta^{\mathrm{I}}, \theta^{\mathrm{I}})$ in $\mathfrak{H}^{\mathrm{I}}$ and a not relabeled subsequence such that*

i) *$(T_\varepsilon u_s, T_\varepsilon v_s, T_\varepsilon \theta_s)$ weakly converges in $H^1_{\Gamma^{\mathcal{M}\mathrm{D}}}(\Omega \setminus S\,;\mathbb{R}^3) \times L^2(\Omega;\mathbb{R}^3) \times L^2(\Omega)$ toward (u, v, θ);*
ii) *$\mathcal{S}^v_\varepsilon v_s$ weakly converges in $L^2(B;\mathbb{R}^3)$ toward v_B if $\mathrm{I_{M1}} = 1$, $\mathcal{S}^\theta_\varepsilon \theta_s$ weakly converges in $L^2(B)$ toward θ_B if $\mathrm{I_{T1}} = 1$;*
iii) *$e^{\mathrm{I_{M2}}}(\varepsilon, \mathcal{S}^{u,\mathrm{I_{M2}}}_\varepsilon u_s)$ weakly converges in $L^2(B;\mathbb{R}^3)$ toward e^u_B when $\mathrm{I_{M2}}$ is positive;*
iv) *$|U^{\mathrm{I}}|^{\mathrm{I}} := \left\{ (u^{\mathrm{I}}, u^{\mathrm{I}})^{\mathrm{I}}_1 + (v^{\mathrm{I}}, v^{\mathrm{I}})^{\mathrm{I}}_2 + (\theta^{\mathrm{I}}, \theta^{\mathrm{I}})^{\mathrm{I}}_3 \right\}^{1/2} \le \underline{\lim}_{s\to\bar s} |U_s|_s$.*

Proof. As

$$|U_s|_s^2 = \int_\Omega a e(T_\varepsilon u_s) \cdot e(T_\varepsilon u_s) + \beta |v_s|^2 + \rho \theta_s^2 \, dx \\ + \int_B \left(\mu_L / r^{\mathrm{M}}_{\mathrm{I_{M2}}}\right) a_L e^{\mathrm{I_{M2}}}(\varepsilon, \mathcal{S}^{u,\mathrm{I_{M2}}}_\varepsilon u_s) \cdot e^{\mathrm{I_{M2}}}(\varepsilon, \mathcal{S}^{u,\mathrm{I_{M2}}}_\varepsilon u_s) + \varepsilon \rho_L |\mathcal{S}^v_\varepsilon v_s|^2 + \varepsilon \beta_L |\mathcal{S}^\theta_\varepsilon \theta_s|^2 \, dx, \tag{3.11}$$

point ii), and point i) *when* $\mathcal{H}_2(\Gamma^{\mathcal{M}\mathrm{D}-}) > 0$ are obvious.

Anyway the boundedness of $(u_s, u_s)_{1,s}$ implies that there exists u^+_Ω in $H^1_{\Gamma^{\mathcal{M}\mathrm{D}+}}(\Omega^+;\mathbb{R}^3)$ and a not relabeled subsequence such that $\big((T_\varepsilon u_s)^+, \gamma_S((T_\varepsilon u_s)^+)\big)$ converges weakly in $H^1(\Omega^+;\mathbb{R}^3)$ and strongly in $L^2(S\,;\mathbb{R}^3)$ toward $\big(u^+_\Omega, \gamma_S(u^+_\Omega)\big)$, respectively. By using Korn inequality and a cut-off function η such that $\eta(x_3) = 1$ if $0 \le x_3 \le L/3$, $\eta(x_3) = 0$ if $x_3 \ge 2L/3$, with $L = \max\{x_3;\ (\widehat{x}, x_3) \in \partial\Omega^+\}$, one has:

$$\Big| [\![T_\varepsilon u_s]\!] \Big|^2_{L^2(S\,;\mathbb{R}^3)} \le C \left(\frac{2\varepsilon}{\mu_L} \mu_L \int_{B_\varepsilon} |e(u_s)|^2 \, dx + \varepsilon \right), \tag{3.12}$$

$$\int_{B_\varepsilon} |u_s|^2 \, dx \le C \left(\varepsilon \left| \gamma_S((T^\varepsilon u_s)^+) \right|^2_{L^2(S\,;\mathbb{R}^3)} + \frac{\varepsilon^2}{\mu_L} \mu_L \int_{B_\varepsilon} |e(u_s)|^2 \, dx + \varepsilon^2 \right). \tag{3.13}$$

Hence assumption (H2) implies that there exist u^-_Ω in $H^1_{\Gamma^{\mathcal{M}\mathrm{D}-}}(\Omega^-;\mathbb{R}^3)$ and a not relabeled subsequence such that $(T_\varepsilon u_s)^-$ converges weakly in $H^1(\Omega^-;\mathbb{R}^3)$ toward u^-_Ω and

$$\int_{B_\varepsilon} |u_s|^2 \, dx \le C \left(\varepsilon + \frac{\varepsilon^2}{\mu_L} \right). \tag{3.14}$$

So, if $\mathrm{I_{M2}} = 1, 2$, $\left(\mathcal{S}^{u,\mathrm{I_{M2}}}_\varepsilon u_s, (\mu_L / \varepsilon^{r^{\mathrm{M}}_{\mathrm{I_{M2}}}}) e^{\mathrm{I_{M2}}}(\varepsilon, \mathcal{S}^{u,\mathrm{I_{M2}}}_\varepsilon u_s) \right)$ is bounded in $L^2(B;\mathbb{R}^3 \times \mathbb{S}^3)$ so that the convergence up to a not relabeled subsequence of $\mathcal{S}^{u,\mathrm{I_{M2}}}_\varepsilon u_s$ in the sense of distributions on B yields that there exists a unique u_B in

$H_{\partial_3}(B;\mathbb{R}^3)$ such that $\left(\widehat{\mathcal{S}_\varepsilon^{u,\mathrm{I_{M2}}}u_s}, (\mathcal{S}_\varepsilon^{u,\mathrm{I_{M2}}}u_s)_3, e^{\mathrm{I_{M2}}}(\varepsilon, \mathcal{S}_\varepsilon^{u,\mathrm{I_{M2}}}u_s)\right)$ weakly converges to $(\widehat{u}_B, u_{B3}, \partial_3 u_B \otimes_S e_3)$ in $L^2(B;\mathbb{R}^2)\times H_{\partial_3}(B)\times L^2(B;\mathbb{S}^3)$ with $\gamma_{S^\pm}(u_B) = \gamma_S(u_\Omega^\pm)$ because $\gamma_{S^\pm}(\mathcal{S}_\varepsilon^{u,\mathrm{I_{M2}}}u_s) = \gamma_S((T_\varepsilon u_s)^\pm)$. Moreover when $\mathrm{I_{M2}} = 2$, one has $e_B^u = 0$ and u_Ω in $H^1(\Omega;\mathbb{R}^3)$.

When $\mathrm{I_{M2}} = 3$, one only has $\left(\widehat{\mathcal{S}_\varepsilon^{u,3}}u_s, e^3(\varepsilon, \mathcal{S}_\varepsilon^{u,3}u_s)\right)$ bounded in $L^2(B;\mathbb{R}^2\times\mathbb{S}^3)$, so that by using the space of infinitesimal rigid displacements it is routine to establish that there exists some u_B in $V_{\mathrm{KL}}(B)$ and e_B^u in $L^2(B;\mathbb{S}^3)$ such that $\left(\mathcal{S}_\varepsilon^{u,3}u_\Omega, e^3(\varepsilon, \mathcal{S}_\varepsilon^{u,3}u_\Omega)\right)$ weakly converges toward (u_B, e_B^u) in $\left(H^1(B;\mathbb{R}^2)\times H^1(B)/\mathbb{R}\right)\times L^2(B;\mathbb{S}^3)$ with $\widehat{e_B^u} = \widehat{e(u_B)}$. Then the perfect contact condition on B_ε, of which one can exploit $\gamma_{S^\pm}(\widehat{\mathcal{S}_\varepsilon^{u,3}u_s}) = \gamma_S(\widehat{(T^\varepsilon u_s)^\pm})$ only, and (3.12) yield $[\![u_\Omega]\!] = 0$, $\gamma_{S^\pm}(\widehat{u}_B) = \gamma_S(\widehat{u_\Omega^\pm}) = \gamma_S(\widehat{u_\Omega})$, which imply $u_\Omega \in H^1(\Omega;\mathbb{R}^3)$, $u_B^M = \gamma_S(\widehat{u_\Omega})$, $\widehat{\nabla} u_B^F = 0$, $\gamma_S(\widehat{u}_\Omega) \in H^1(S;\mathbb{R}^2)$. Moreover as there exists a constant C_ε such that $\gamma_S((T_\varepsilon u_s)_3^+/\varepsilon) + C_\varepsilon$ strongly converges in $L^2(S)$ toward $\gamma_{S^+}(u_B^F)$, one deduces that $\gamma_S(u_{\Omega 3})$ is constant.

Proceeding as in the previous case, we deduce, when $\mathrm{I_{M2}} = 4$, that there exists u_B in $V_{\mathrm{KL}}(B)$ with $u_B^F = \gamma_S(u_{\Omega 3})$ such that $(\mathcal{S}_\varepsilon^{u,4}u_s, e^4(\varepsilon, \mathcal{S}_\varepsilon^{u,4}u_s))$ weakly converges in $H^1(B;\mathbb{R}^3)\times L^2(B;\mathbb{S}^3)$ toward (u_B, e_B^u) with $\widehat{e_B^u} = \widehat{e(u_B)}$. □

3.1.2. The limit operator A^{I}

According to Trotter's theory of approximation of semi-groups of linear operators acting on sequences of variable Hilbert spaces ([13, 14]), we examine the asymptotic behavior of the resolvent $(I - A_s)^{-1}$ in order to guess the limit operator A^{I}. Proposition 2.1 implies that a sequence $U_s = (u_s, v_s, \theta_s)$ such that $|U_s|_s + |A_s U_s|_s \le C$ involves $(v_s, v_s)_{1,s} + (\theta_s, \theta_s)_{4,s} \le C$ *in addition to* $|U_s|_s \le C$ that we already considered. For this purpose we introduce the following spaces $G_\theta^{\mathrm{I_{T2}}}$ of temperatures and operators g_B^θ:

$$\begin{cases} G_\theta^0 := \left\{ \theta^{\mathrm{I}} = (\theta_\Omega, \theta_B) \in H^1(\Omega^0)\times L^2(B) \right\}, \quad g_B^\theta := 0; \\ G_\theta^1 := \left\{ \theta^{\mathrm{I}} = (\theta_\Omega, \theta_B) \in H^1(\Omega^1)\times H_{\partial_3}(B);\ \gamma_{S^\pm}(\theta_B) = \gamma_S(\theta_\Omega^\pm) \right\}, \quad g_B^\theta := (0, \partial_3\theta_B); \\ G_\theta^2 := \left\{ \theta^{\mathrm{I}} = (\theta_\Omega, \theta_B) \in H^1(\Omega)\times H_{\partial_3}(B);\ \partial_3\theta_B = 0, \gamma_{S^\pm}(\theta_B) = \gamma_S(\theta_\Omega) \right\}, \quad g_B^\theta := 0; \\ G_\theta^3 := \left\{ \theta^{\mathrm{I}} = (\theta_\Omega, \theta_B) \in H^1(\Omega)\times H^1(B);\ \gamma_S(\theta_\Omega) \in H^1(S), \partial_3\theta_B = 0, \gamma_{S^\pm}(\theta_B) = \gamma_S(\theta_\Omega) \right\}, \\ \qquad\qquad g_B^\theta := (\widehat{\nabla}\gamma_S\theta_\Omega, 0); \\ G_\theta^4 := \left\{ \theta^{\mathrm{I}} = (\theta_\Omega, \theta_B) \in H^1(\Omega)\times H^1(B);\ \gamma_S(\theta_\Omega) = C (= 0 \text{ if } \mathcal{H}_1(\gamma^{\mathcal{T}\mathrm{D}}) > 0), \right. \\ \qquad\qquad \left. \partial_3\theta_B = 0, \gamma_{S^\pm}(\theta_B) = \gamma_S(\theta_\Omega) \right\}, \quad g_B^\theta := 0. \end{cases} \tag{3.15}$$

Thus, if for all θ_B in $H^1(B)$

$$g^{\mathrm{I_{T2}}}(\varepsilon, \theta_B) = \begin{cases} \left(\varepsilon\widehat{\nabla}\theta_B, \partial_3\theta_B\right) & \text{when } \mathrm{I_{M2}} \le 2, \\ \left(\widehat{\nabla}\theta_B, \partial_3\theta_B/\varepsilon\right) & \text{when } \mathrm{I_{M2}} > 2, \end{cases} \tag{3.16}$$

we have the obvious scalar variant of Proposition 3.1:

Proposition 3.2. *For all sequences such that $(\theta_s, \theta_s)_{3,s} \le C$, there exist θ^{I} in $G_\theta^{\mathrm{I_{T2}}}$ such that*

i) $T_\varepsilon\theta_s$ weakly converges in $H^1_{\Gamma^{\mathcal{T}\mathrm{D}}}(\Omega^{\mathrm{I_{T2}}})$ toward θ_Ω;
ii) $g^{\mathrm{I_{T2}}}(\varepsilon, \mathcal{S}_\varepsilon^\theta\theta_s)$ converges weakly in $L^2(B;\mathbb{R}^3)$ toward g_B^θ, when $\mathrm{I_{M2}} \ge 1$.

Note that it is only when $\mathrm{I_{T2}} \le 1$, that the "additional" temperature θ_B depends on x_3. We therefore make a supplementary assumption. Let $\bar{\alpha}_L^{\mathrm{I}}$ in $[0,+\infty)$ satisfying:

$$\begin{cases} \mathrm{I}_{\mathrm{M2}}=0 & \text{no condition,} \\ \mathrm{I}_{\mathrm{M2}}=1 & \bar{\alpha}_L^{\mathrm{I}} := \lim_{s\to\bar{s}} \varepsilon\alpha_L, \\ \mathrm{I}_{\mathrm{M2}}=2 & \text{no condition,} \\ \mathrm{I}_{\mathrm{M2}}=3 & \bar{\alpha}_L^{\mathrm{I}} := \lim_{s\to\bar{s}} \alpha_L, \\ \mathrm{I}_{\mathrm{M2}}=4 & \bar{\alpha}_L^{\mathrm{I}} := \lim_{s\to\bar{s}} \alpha_L/\varepsilon, \end{cases} \tag{H3}$$

it is reasonable to suggest the following forms $(\cdot,\cdot)_4^{\mathrm{I}}$ and $(\cdot,\cdot)_5^{\mathrm{I}}$ as "potential limits" of $(\cdot,\cdot)_{4,s}$ and $(\cdot,\cdot)_{5,s}$:

$$\begin{aligned} (\theta^{\mathrm{I}},\theta'^{\mathrm{I}})_4^{\mathrm{I}} &:= \int_\Omega \kappa\nabla\theta_\Omega\cdot\nabla\theta'_\Omega\,dx + \bar{\kappa}_L^{\mathrm{I}}\int_B g_B^\theta\cdot g_B^{\theta'}\,dx \quad \forall\theta^{\mathrm{I}},\theta'^{\mathrm{I}}\in G_\theta^{\mathrm{I}}, \\ (u^{\mathrm{I}},\theta^{\mathrm{I}})_5^{\mathrm{I}} &:= \int_\Omega (ae(u)\cdot\alpha)\theta_\Omega\,dx + (\bar{\mu}_L^{\mathrm{I}}\bar{\alpha}_L^{\mathrm{I}})\int_B (a_Le_B^u\cdot I)\theta_B\,dx \quad \forall u^{\mathrm{I}}\in\mathfrak{H}_u^{\mathrm{I}},\forall\theta^{\mathrm{I}}\in G_\theta^{\mathrm{I}}. \end{aligned} \tag{3.17}$$

It will be convenient to introduce for all θ^{I} in G_θ^{I}

$$\check{\theta}^{\mathrm{I}} := \begin{cases} \theta_\Omega & \text{if } \mathrm{I}_{\mathrm{T1}}=0, \\ (\theta_\Omega,\theta_B) & \text{if } \mathrm{I}_{\mathrm{T1}}=1. \end{cases} \tag{3.18}$$

The boundedness of *both* $\rho_L\int_{B_\varepsilon}|v_s|^2\,dx$ and $\mu_L\int_{B_\varepsilon} ae(v_s)\cdot e(v_s)\,dx$ leads us, when $\mathrm{I}_{\mathrm{M1}}=1$ and $\mathrm{I}_{\mathrm{M2}}=3,4$, to introduce a special space $\mathfrak{S}\mathfrak{H}_v^{\mathrm{I}}$ for velocities and consequently a special space of limit possible states with finite energy $\mathfrak{S}\mathfrak{H}^{\mathrm{I}}$.
Let

$$\begin{aligned} L_{\mathrm{KL}}^M(B) &:= \left\{ v\in L^2(B;\mathbb{R}^3) \text{ s.t. } v(x)=(\widehat{v}(\widehat{x}),0), \widehat{v}\in L^2(S;\mathbb{R}^2) \right\}, \\ L_{\mathrm{KL}}^F(B) &:= \left\{ v\in H^{-1}(B;\mathbb{R}^3) \text{ s.t. } v(x)=(-x_3\widehat{\nabla}v_3,v_3), v_3\in L^2(S) \right\}, \end{aligned} \tag{3.19}$$

then

$$\mathfrak{S}\mathfrak{H}_v^{\mathrm{I}} := \begin{cases} \mathfrak{H}_v^{\mathrm{I}} & \text{if } \mathrm{I}_{\mathrm{M1}}=0 \text{ or } \mathrm{I}_{\mathrm{M2}}\le 2; \\ \left\{ (v_\Omega,v_B)\in L^2(\Omega;\mathbb{R}^3)\times L_{\mathrm{KL}}^M(B) \right\} & \text{if } \mathrm{I}_{\mathrm{M1}}=1 \text{ and } \mathrm{I}_{\mathrm{M2}}=3; \\ \left\{ (v_\Omega,v_B)\in L^2(\Omega;\mathbb{R}^3)\times L_{\mathrm{KL}}^F(B) \right\} & \text{if } \mathrm{I}_{\mathrm{M1}}=1 \text{ and } \mathrm{I}_{\mathrm{M2}}=4, \end{cases} \tag{3.20}$$

$$(v^{\mathrm{I}},v'^{\mathrm{I}})_2^{\mathrm{I}} := (\mathring{v}^{\mathrm{I}},\mathring{v}'^{\mathrm{I}})_2^{\mathrm{I}} \quad \forall v^{\mathrm{I}},v'^{\mathrm{I}}\in\mathfrak{S}\mathfrak{H}_v^{\mathrm{I}}, \tag{3.21}$$

where

$$\mathring{v}^{\mathrm{I}} = \begin{cases} v_\Omega & \text{if } \mathrm{I}_{\mathrm{M1}}=0; \\ (v_\Omega,v_B) & \text{if } \mathrm{I}_{\mathrm{M1}}=1 \text{ and } \mathrm{I}_{\mathrm{M2}}\le 3; \\ (v_\Omega,(0,v_{B3})) & \text{if } \mathrm{I}_{\mathrm{M1}}=1 \text{ and } \mathrm{I}_{\mathrm{M2}}=4, \end{cases} \tag{3.22}$$

and

$$\mathfrak{S}\mathfrak{H}^{\mathrm{I}} := \mathfrak{H}_u^{\mathrm{I}}\times\mathfrak{S}\mathfrak{H}_v^{\mathrm{I}}\times\mathfrak{H}_\theta^{\mathrm{I}} \tag{3.23}$$

equipped with the norm $|\cdot|^{\mathrm{I}}$. Thus, in order to proceed in a unitary manner, we denote the space of admissible virtual generalized velocities and temperatures by Z^{I}:

$$Z^{\mathrm{I}} := \left\{ (v^{\mathrm{I}}, \theta^{\mathrm{I}}) \in \mathfrak{S}\mathfrak{H}_u^{\mathrm{I}} \times G_\theta^{\mathrm{I_{T2}}} \right\}, \tag{3.24}$$

$$\mathfrak{S}\mathfrak{H}_u^{\mathrm{I}} := \begin{cases} \left\{ u^{\mathrm{I}} \in \mathfrak{H}_u^{\mathrm{I}} \text{ with } u_B \text{ any element of } L^2(B;\mathbb{R}^3)) \right\} & \text{when } \mathrm{I_{M2}} = 0, \\ \mathfrak{H}_u^{\mathrm{I}} & \text{when } \mathrm{I_{M2}} > 0, \end{cases} \tag{3.25}$$

where, for all elements of $\mathfrak{S}\mathfrak{H}_u^{\mathrm{I}}$, $\check{u}^{\mathrm{I}}$ is defined by:

$$\check{u}^{\mathrm{I}} := \begin{cases} u_\Omega & \text{if } \mathrm{I_{M1}} = 0; \\ (u_\Omega, u_B) & \text{if } \mathrm{I_{M1}} = 1, \end{cases} \tag{3.26}$$

we are in a position to define operator A^{I} in $\mathfrak{S}\mathfrak{H}^{\mathrm{I}}$ by:

$$\begin{cases} D(A^{\mathrm{I}}) := \left\{ U^{\mathrm{I}} = (u^{\mathrm{I}}, v^{\mathrm{I}}, \theta^{\mathrm{I}}) \in \mathfrak{S}\mathfrak{H}^{\mathrm{I}}; \begin{cases} \text{i)} & \exists!\, (\widetilde{v}^{\mathrm{I}}, \widetilde{\theta}^{\mathrm{I}}) \in Z^{\mathrm{I}} \text{ s.t. } \check{\widetilde{v}}^{\mathrm{I}} = v^{\mathrm{I}}, \\ \text{ii)} & \exists!\, (w^{\mathrm{I}}, \tau^{\mathrm{I}}) \in \mathfrak{S}\mathfrak{H}_v^{\mathrm{I}} \times \mathfrak{H}_\theta^{\mathrm{I}} \text{ s.t.} \\ & \quad (\mathring{w}^{\mathrm{I}}, \check{v}'^{\mathrm{I}})_2^{\mathrm{I}} + (u^{\mathrm{I}}, v'^{\mathrm{I}})_1^{\mathrm{I}} - (v'^{\mathrm{I}}, \widetilde{\theta}^{\mathrm{I}})_5^{\mathrm{I}} = 0, \\ & \quad (\tau^{\mathrm{I}}, \check{\theta}'^{\mathrm{I}})_3^{\mathrm{I}} + (\widetilde{\theta}^{\mathrm{I}}, \theta'^{\mathrm{I}})_4^{\mathrm{I}} + (\widetilde{v}^{\mathrm{I}}, \theta'^{\mathrm{I}})_5^{\mathrm{I}} = 0, \quad \forall (v'^{\mathrm{I}}, \theta'^{\mathrm{I}}) \in Z^{\mathrm{I}}, \end{cases} \right\} \\ A^{\mathrm{I}} U^{\mathrm{I}} = (\widetilde{v}^{\mathrm{I}}, \mathring{w}^{\mathrm{I}}, \tau^{\mathrm{I}}). \end{cases} \tag{3.27}$$

Similar to the case of A_s, it can be checked easily that A^{I} is m-dissipative and, more specifically, that for all $\phi = (\phi^1, \phi^2, \phi^3)$ in $\mathfrak{S}\mathfrak{H}^{\mathrm{I}}$:

$$\begin{cases} \overline{U}^{\mathrm{I}} = (\overline{u}^{\mathrm{I}}, \overline{v}^{\mathrm{I}}, \overline{\theta}^{\mathrm{I}}) \text{ s.t.} \\ \overline{U}^{\mathrm{I}} - A^{\mathrm{I}} \overline{U}^{\mathrm{I}} = \phi \end{cases} \iff \begin{cases} \overline{u}^{\mathrm{I}} = \widetilde{v}^{\mathrm{I}} + \phi^1, \\ \overline{v}^{\mathrm{I}} = \widetilde{v}_\Omega \quad \text{if } \mathrm{I_{M1}} = 0, \quad \overline{v}^{\mathrm{I}} = (\widetilde{v}_\Omega, \widetilde{v}_B) \quad \text{if } \mathrm{I_{M1}} = 1, \\ \overline{\theta}^{\mathrm{I}} = \widetilde{\theta}_\Omega \quad \text{if } \mathrm{I_{T1}} = 0, \quad \overline{\theta}^{\mathrm{I}} = (\widetilde{\theta}_\Omega, \widetilde{\theta}_B) \quad \text{if } \mathrm{I_{T1}} = 1, \\ \overline{z}^{\mathrm{I}} = (\widetilde{v}^{\mathrm{I}}, \widetilde{\theta}^{\mathrm{I}}) \in Z^{\mathrm{I}}; \quad \Psi^{\mathrm{I}}(\overline{z}^{\mathrm{I}}, z) = \mathcal{L}^{\mathrm{I}}(z) \quad \forall z \in Z^{\mathrm{I}} \text{ with} \\ \Psi^{\mathrm{I}} := \Phi^{\mathrm{I}} + K^{\mathrm{I}}, \\ \Phi^{\mathrm{I}}(z, z') := (v, v')_1^{\mathrm{I}} + (\theta, \theta')_4^{\mathrm{I}} - (v', \theta)_5^{\mathrm{I}} + (v, \theta')_5^{\mathrm{I}} \\ K^{\mathrm{I}}(\check{z}, \check{z}') := (\check{v}, \check{v}')_2^{\mathrm{I}} + (\check{\theta}, \check{\theta}')_3^{\mathrm{I}}, \quad \forall z = (v, \theta), \forall z' = (v', \theta') \in Z^{\mathrm{I}}, \\ \mathcal{L}^{\mathrm{I}}(z) := -(\phi^1, v)_1^{\mathrm{I}} + K^{\mathrm{I}}((\phi^2, \phi^3), z). \end{cases} \tag{3.28}$$

Consequently, the same statement as that of Theorem 2.1 is valid for the following equation, which will be shown to describe the asymptotic behavior of the solution to $(\mathcal{P}_s)$:

$$(\mathcal{P}^{\mathrm{I}}) \quad \begin{cases} \dfrac{dU^{\mathrm{I}r}}{dt} - A^{\mathrm{I}} U^{\mathrm{I}r} = F^{\mathrm{I}} := \left(u^{\mathrm{I}e} - \dfrac{du^{\mathrm{I}e}}{dt}, -\dfrac{du^e}{dt} + \dfrac{f^{\mathrm{I}}}{\rho}, -\dfrac{d\theta^{\mathrm{I}e}}{dt} \right), \\ U^{\mathrm{I}r}(0) = U^{\mathrm{I}r,0}, \end{cases} \tag{3.29}$$

where $f^{\mathrm{I}} = f$ if $\mathrm{I_{M1}} = 0$, $f^{\mathrm{I}} = (f, 0)$ if $\mathrm{I_{M1}} = 1$ with

$$\begin{aligned} & z^{\mathrm{I}e} = (u^{\mathrm{I}e}, \theta^{\mathrm{I}e}) \in Z^{\mathrm{I}}; \quad \Phi^{\mathrm{I}}(z^{\mathrm{I}e}(t), z) = L(t)(z) \quad \forall z \in Z^{\mathrm{I}}\ \forall t \in [0, T], \\ & L(t)(z) := \int_{\Gamma^{\mathcal{M}\mathrm{N}}} g^{\mathcal{M}} \cdot v_\Omega \, d\mathcal{H}_2 + \int_{\Gamma^{\mathcal{T}\mathrm{N}}} g^{\mathcal{T}} \theta_\Omega \, d\mathcal{H}_2. \end{aligned} \tag{3.30}$$

We set

$$U^{\mathrm{I}e} = \left(u^{\mathrm{I}e}, z^{\mathrm{I}e}\right), \quad U^{\mathrm{I}} = U^{\mathrm{I}e} + U^{\mathrm{I}r}. \tag{3.31}$$

3.2. Convergence

To prove the convergence of the solution U_s to $(\mathcal{P}_s)$ toward the solution U^{I} to $(\mathcal{P}^{\mathrm{I}})$, we use the framework of Trotter's theory of approximation of semi-groups of linear operators acting on *variable spaces* (see [13, 14]) because U_s and U^{I} *do not inhabit the same space*.

First let the representation operator P_s^{I} defined by

$$U \in \Im\mathfrak{H}^{\mathrm{I}} \mapsto P_s^{\mathrm{I}} U = (u_s^*, v_s^*, \theta_s^*) \in \mathfrak{H}_s \tag{3.32}$$

with

- $u_s^* \in H^1_{\Gamma_\varepsilon^{\mathcal{M}\mathrm{D}}}(O_\varepsilon; \mathbb{R}^3)$;

$$(u_s^*, u_s')_{1,s} = \int_\Omega ae(u_\Omega) \cdot e(T_\varepsilon u_s')\, dx + \left(\mu_L / \varepsilon^{r^{\mathrm{M}}_{\mathrm{I_{M2}}}}\right) \int_B a_L e_B^u \cdot e(\varepsilon, \mathcal{S}_\varepsilon^{u,\mathrm{I_{M2}}} u_s')\, dx \quad \forall u_s' \in H^1_{\Gamma_\varepsilon^{\mathcal{M}\mathrm{D}}}(O_\varepsilon; \mathbb{R}^3), \tag{3.33}$$

- $(v_s^*, \theta_s^*) \in L^2(O_\varepsilon; \mathbb{R}^3) \times L^2(O_\varepsilon)$;

$$(v_s^*, v_s')_{2,s} := \int_\Omega \rho v_\Omega \cdot T_\varepsilon v_s'\, dx + \varepsilon \rho_L \int_B v_B \cdot \mathcal{S}_\varepsilon^v v_s'\, dx \quad \forall v_s' \in L^2(O_\varepsilon; \mathbb{R}^3), \tag{3.34}$$

$$(\theta_s^*, \theta_s')_{3,s} := \int_\Omega \beta \theta_\Omega T_\varepsilon \theta_s'\, dx + \varepsilon \beta_L \int_B \theta_B \mathcal{S}_\varepsilon^\theta \theta_s'\, dx \quad \forall \theta_s' \in L^2(O_\varepsilon), \tag{3.35}$$

(of course when $\mathrm{I_{M1}} = 0$ or $\mathrm{I_{T1}} = 0$, we set $v_B = 0$ or $\theta_B = 0$) and which satisfies

Proposition 3.3.

i) *There exists a positive constant C such that $|P_s^{\mathrm{I}} U|_s \le C |U^{\mathrm{I}}|^{\mathrm{I}}$ for all U^{I} in $\Im\mathfrak{H}^{\mathrm{I}}$ and all s;*

ii) *When s tends to $\bar{s}$, $|P_s^{\mathrm{I}} U|_s$ converges toward $|U|^{\mathrm{I}}$ for all U^{I} in $\Im\mathfrak{H}^{\mathrm{I}}$.*

Proof. The part of the result concerning u_s^* when $\mathrm{I_{M2}} = 0$, v_s^* and θ_s^* is obvious. By choosing $u_s' = u_s^*$ and arguing as in the proof of Proposition 3.1, there exist $(u_\Omega^*, e_B^{u^*})$ in $\mathfrak{H}_u^{\mathrm{I}}$ such that $\left(T_\varepsilon u_s^*, e^{\mathrm{I_{M2}}}(\varepsilon, \mathcal{S}_\varepsilon^{u,\mathrm{I_{M2}}} u_s^*)\right)$ weakly converges (up to a not relabeled subsequence) toward $(u_\Omega^*, e_B^{u^*})$ in $H^1_{\Gamma^{\mathcal{M}\mathrm{D}}}(\Omega^{\mathrm{I_{M2}}}; \mathbb{R}^3) \times L^2(B; \mathbb{S}^3)$. To identify $u^* := (u_\Omega^*, e_B^{u^*})$ as $u := (u_\Omega, e_B^u)$, it suffices, for all $(u_\Omega', e_B^{u'})$ in $\mathfrak{H}_u^{\mathrm{I}}$ with $e_B^{u'}$ in $C^\infty(\overline{\Omega}; \mathbb{S}^3)$, to build a sequence u_s' in $H^1_{\Gamma^{\mathcal{M}\mathrm{D}}}(O_\varepsilon; \mathbb{R}^3)$ such that :

$$\begin{aligned}
&\int_{\Omega_\varepsilon} a_\varepsilon e(u_\varepsilon^*) \cdot e(u_s')\, dx \to \int_\Omega ae(u_\Omega^*) \cdot e(u_\Omega')\, dx,\\
&\int_{\Omega_\varepsilon} a_\varepsilon e(u_\varepsilon) \cdot e(T_\varepsilon u_s')\, dx \to \int_\Omega ae(u_\Omega) \cdot e(u_\Omega')\, dx,\\
&\left(\mu_L / \varepsilon^{r^{\mathrm{M}}_{\mathrm{I_{M2}}}}\right) \int_B a_L e_B^u \cdot e^{\mathrm{I_{M2}}}(\varepsilon, \mathcal{S}_\varepsilon^{u,\mathrm{I_{M2}}} u_s')\, dx \to \bar{\mu}_L \int_B a_L e_B^u \cdot e_B^{u'}\, dx,\\
&\mu_L \int_{B_\varepsilon} a_L e(u_s^*) \cdot e(u_s')\, dx \to \bar{\mu}_L \int_B a_L e_B^{u^*} \cdot e_B^{u'}\, dx,
\end{aligned} \tag{3.36}$$

Clearly, when $\mathrm{I_{M2}} = 1, 2$, $u_s' := \begin{cases} \left(\mathcal{S}_\varepsilon^{u,\mathrm{I_{M2}}}\right)^{-1} u_B' & \text{in } B_\varepsilon \\ u_\Omega'(\cdot \mp \varepsilon e_3) & \text{in } \Omega_\varepsilon^\pm \end{cases}$ satisfies (3.36). When $\mathrm{I_{M2}} = 3, 4$, we use a trick of the mathematical derivation of Kirchhoff-Love theory of plates [15, 16]. Let φ_s, ψ_s defined as follows:

$$
\begin{aligned}
&\varphi_s := \begin{cases} \gamma_S(u'_\Omega) \quad \text{in } B_\varepsilon & \text{if } \mathrm{I_{M2}} = 3,\\ \left(\mathcal{S}^{u,4}_\varepsilon\right)^{-1} u'_B = \left(\varepsilon u'^M_B - x_3 \widehat{\nabla} \gamma_S(u'_{\Omega 3}), \gamma_S(u'_{\Omega 3})\right) & \text{if } \mathrm{I_{M2}} = 4,\end{cases}\\
&\psi_s := \left(\mathcal{S}^{u,\mathrm{I_{M2}}}_\varepsilon\right)^{-1} \psi_{B\varepsilon}, \quad \psi_{B\varepsilon}(x) := \varepsilon \int_0^{x_3} \left[\widehat{w}(\widehat{x},\zeta) - \varepsilon \int_0^\zeta \widehat{\nabla} w_3(\widehat{x},\xi)\, d\xi\right] d\zeta \quad \text{a.e. } x \in B,
\end{aligned}
\tag{3.37}
$$

where $(e^{u'}_B)^\perp = w \otimes_S e_3$, then the field $u'_s := \varphi_s + \psi_s$ is such that $e^{\mathrm{I_{M2}}}(\varepsilon, \mathcal{S}^{u,\mathrm{I_{M2}}}_\varepsilon) u'_s$ converges strongly in $L^2(B; \mathbb{S}^3)$ toward $e^{u'}_B$ and $|\gamma_{S^\pm_\varepsilon}(u'_s - u'_\Omega(\cdot \mp \varepsilon e_3)|_{L^2(S^\pm_\varepsilon;\mathbb{R}^3)} = O(\varepsilon^{\mathrm{I_{M2}}-2})$. Hence u'_s may be extended into Ω_ε to an element of $H^1_{\Gamma^{MD}_\varepsilon}(O_\varepsilon; \mathbb{R}^3)$ still denoted u'_s which satisfies (3.36). Lastly by choosing $u'_s = u^*_s$ in (3.33), one obtains that $(u^*_s, u^*_s)_{1,s}$ tends to $(u,u)^{\mathrm{I}}_1$. □

Next we state that:

$$
U_s \text{ in } \mathfrak{H}_s \textit{ converges in the sense of Trotter toward } U^{\mathrm{I}} \textit{ in } \mathfrak{H}^{\mathrm{I}} \textit{ if } \lim_{s\to\bar{s}} |P^{\mathrm{I}}_s U^{\mathrm{I}} - U_s|_s = 0. \tag{3.38}
$$

Note that U_s converges in the sense of Trotter toward U is equivalent to $T_\varepsilon(u_s, v_s, \theta_s)$ converges strongly toward $(u_\Omega, v_\Omega, \theta_\Omega)$ in $H^1(\Omega^{\mathrm{I_{M2}}}, \mathbb{R}^3) \times L^2(\Omega; \mathbb{R}^3) \times L^2(\Omega)$ and $\left((\mu_L/\varepsilon^{r^{\mathrm{M}}_{\mathrm{I_{M2}}}})^{1/2} e^{\mathrm{I_{M2}}}(\varepsilon, \mathcal{S}^{u,\mathrm{I_{M2}}}_\varepsilon u_s),\ (\rho_L\varepsilon)^{1/2} \mathcal{S}^v_\varepsilon v_s,\ (\beta_L \varepsilon)^{1/2} \mathcal{S}^\theta_\varepsilon \theta_s\right)$ converges strongly toward $(\bar{\mu}^{1/2}_L e^u_B,\ \bar{\rho}^{1/2}_L v_B,\ \bar{\beta}^{1/2}_L \theta_B)$ in $L^2(B; \mathbb{S}^3 \times \mathbb{R}^3 \times \mathbb{R})$.

Lastly we conclude by making an additional assumption (H4) about the initial state and establishing the

Proposition 3.4. *There hold*

i) $\forall \phi \in \mathfrak{H}^{\mathrm{I}}, \quad \lim_{s\to\bar{s}} |P^{\mathrm{I}}_s (I - A^{\mathrm{I}})^{-1}\phi - (I - A_s)^{-1} P^{\mathrm{I}}_s \phi|_s = 0,$

ii) $\lim_{s\to\bar{s}} |P^{\mathrm{I}}_s U^{\mathrm{I}e}(t) - U^{\mathrm{I}e}_s(t)|_s = 0$ *uniformly on* $[0,T]$,

iii) $\lim_{s\to\bar{s}} \int_0^T |P^{\mathrm{I}}_s F^{\mathrm{I}}(t) - F_s(t)|_s\, dt = 0.$

Proof. Taking advantage of Proposition 2.1 and (3.28), it suffices to build for all z' in a dense subspace of Z^{I} a sequence z'_s in H_s such that if U_s satisfies $|U_s|_s \le C$ *and* $(v_s, v_s)_{1,s} + (\theta_s, \theta_s)_{4,s} \le C$ one has

$$
\begin{aligned}
&\lim_{s\to\bar{s}} (v_s, v'_s)_{1,s} = (v, v')^{\mathrm{I}}_1, \quad \lim_{s\to\bar{s}} (\theta_s, \theta'_s)_{4,s} = (\theta, \theta')^{\mathrm{I}}_4,\\
&\lim_{s\to\bar{s}} (v'_s, \theta_s)_{5,s} = (v', \theta)^{\mathrm{I}}_5, \quad \lim_{s\to\bar{s}} (v_s, \theta'_s)_{5,s} = (v, \theta')^{\mathrm{I}}_5,\\
&\lim_{s\to\bar{s}} K_s\left((v_s, \theta_s), (v'_s, \theta'_s)\right) = K^{\mathrm{I}}\left((v,\theta),(v',\theta')\right),
\end{aligned}
\tag{3.39}
$$

where $((v_\Omega, e^v_B), (v_\Omega, v_B), \theta)$ is the limit in the sense of Trotter of (v_s, v_s, θ_s). From Proposition 3.3, the choice $v'_s = v'^*_s$ is in order, while we use θ'_s defined by

$$
\theta'_s(x) = \begin{cases} \theta'_\Omega(x \mp \varepsilon e_3) & \text{a.e. } x \in \Omega^{\mathrm{I}}_\varepsilon,\\ (\mathcal{S}^\varepsilon \theta)^{-1}(\theta'_B) & \text{a.e. } x \in B_\varepsilon,\end{cases} \tag{3.40}
$$

for any $(\theta'_\Omega, \theta'_B)$ in $G^{\mathrm{I_{T2}}}_\theta$ such that θ'_B belongs to $H^1(B)$, because $\left(T_\varepsilon \theta'_s,\ (\kappa_L/\varepsilon^{r^{\mathrm{T}}_{\mathrm{I_{T2}}}}) g^{\mathrm{I_{T2}}}_\varepsilon(\varepsilon, \mathcal{S}^\theta_\varepsilon \theta'_s)\right)$ *strongly* converges in $H^1(\Omega^{\mathrm{I_{T2}}}) \times L^2(B; \mathbb{R}^3)$ toward $\left(\theta'_\Omega, \bar{\kappa}^{\mathrm{I}}_L g^{\theta'}_B\right)$. □

Thus the convergence result can be stated as:

Theorem 3.1. *Under assumptions* (H1)–(H3) *and*

$$
\exists U^{\mathrm{I}0} \in U^{\mathrm{I}e}(0) + D(A^{\mathrm{I}}) \textit{ s.t. } \lim_{s\to\bar{s}} |P^{\mathrm{I}}_s U^{\mathrm{I}0} - U^0_s|_s = 0 \tag{H4}
$$

the solution to

$$\frac{dU_s}{dt} - A_s(U_s - U_s^e) = (0, f_s, 0), \quad U_s(0) = U_s^0 \tag{3.41}$$

converges toward the solution to

$$\frac{dU^{\rm I}}{dt} - A^{\rm I}(U^{\rm I} - U^{{\rm I}e}) = (0, f^{\rm I}/\rho, 0), \quad U^{\rm I}(0) = U^{\rm I0} \tag{3.42}$$

in the sense $\lim_{s\to\bar{s}} |P_s^{\rm I} U^{\rm I}(t) - U_s(t)|_s^{\rm I} = 0$, $\lim_{s\to\bar{s}} |U_s(t)|_s = |U^{\rm I}(t)|^{\rm I}$ *uniformly on* $[0, T]$.

4. Thermomechanical presentation of the results

Here we intend to make more explicit the formulation (3.42) of the limit behavior of the structure. An over dot $'\,\dot{}\,'$ denoting differentiation with respect to time, the motion equation reads as:

$$\int_\Omega \rho \ddot{u}_\Omega \cdot v'_\Omega + a\left(e(u_\Omega) - \theta_\Omega \alpha I\right) \cdot e(v'_\Omega))\, dx + \int_B \bar{\rho}_L \ddot{\mathring{u}}_B \cdot v'_B + \bar{\mu}_L^{\rm I} a_L^{\rm I}\left(e_B^u - \theta_B \bar{\alpha}_L^{\rm I} I\right) \cdot e_B^{v'}\, dx$$
$$= \int_\Omega f \cdot v'_\Omega\, dx + \int_{\Gamma^{\mathcal{M}{\rm N}}} g^{\mathcal{M}} \cdot v'_\Omega\, d\mathcal{H}_2, \quad \forall v' \in \mathfrak{S}\mathfrak{H}_v^{\rm I} \tag{4.1}$$

while the "energy" equation reads as:

$$\int_\Omega \beta \dot{\theta}_\Omega \theta'_\Omega + \kappa \nabla\theta_\Omega \cdot \nabla\theta'_\Omega + \left(a e(\dot{u}_\Omega) \cdot \alpha I\right)\theta'_\Omega\, dx + \int_B \bar{\beta}_L \dot{\theta}_B \theta'_B + \bar{\kappa}_L^{\rm I} g_B^\theta \cdot g_B^{\theta'} + \bar{\mu}_L^{\rm I} \bar{\alpha}_L^{\rm I}\left(a_L^{\rm I}(e_B^{\dot{u}}) \cdot I\right)\theta'_B\, dx$$
$$= \int_{\Gamma^{\mathcal{T}{\rm N}}} g^{\mathcal{T}} \theta'_\Omega\, d\mathcal{H}_2, \quad \forall \theta' \in G_\theta^{\rm I_{T2}}. \tag{4.2}$$

Clearly the stress $\sigma_\Omega^\pm$, thermal flux $q_\Omega^\pm$, displacement and temperature fields in the adhering bodies that occupy Ω^+ and Ω^- satisfy the following relations written in strong form:

$$\begin{cases} \rho \ddot{u}_\Omega^\pm - \ \mathrm{div}\ \sigma_\Omega^\pm = f \text{ in } \Omega^\pm, \quad \sigma_\Omega^\pm n = g^{\mathcal{M}\pm} \text{ on } \Gamma^{\mathcal{M}{\rm N}\pm} \\ \sigma_\Omega^\pm = a\big(e(u_\Omega^\pm) - \theta_\Omega^\pm \alpha I\big) \text{ in } \Omega^\pm \\ \beta \dot{\theta}_\Omega^\pm - \ \mathrm{div}\ q_\Omega^\pm + a e(\dot{u}_\Omega^\pm) \cdot \alpha I = 0 \text{ in } \Omega^\pm, \quad q_\Omega^\pm \cdot n = g^{\mathcal{T}\pm} \text{ on } \Gamma^{\mathcal{T}{\rm N}\pm} \\ q_\Omega^\pm = \kappa \nabla \theta_\Omega^\pm \text{ in } \Omega^\pm \end{cases} \tag{4.3}$$

where n denotes the outward normal to Ω, *together with a thermomechanical contact condition along* S, the common boundary of Ω^+ and Ω^-. This corresponds to the transient response to the loading $(f, g^{\mathcal{M}}, g^{\mathcal{T}})$ of each adhering body clamped on $\Gamma^{\mathcal{M}{\rm D}\pm}$ maintained at a uniform temperature T_0 on $\Gamma^{\mathcal{T}{\rm D}\pm}$ and thermomechanically linked along S. These contact conditions, *which stem from the limit behavior of the adhesive layer*, can be deduced from the various expressions of the two integrals on B in (4.1)-(4.2). The motion and "energy" equations will be formulated in the form:

$$\mathcal{M}_\Omega(v'_\Omega) + \mathcal{M}_B(v'_B) = \int_\Omega f \cdot v'_\Omega\, dx + \int_{\Gamma^{\mathcal{M}{\rm N}}} g^{\mathcal{M}} \cdot v'_\Omega\, d\mathcal{H}_2, \quad \forall v' \in V^{\rm I_{M2}} \tag{4.4}$$

$$\mathcal{T}_\Omega(\theta'_\Omega) + \mathcal{T}_B(\theta'_B) = \int_{\Gamma^{\mathcal{T}{\rm N}}} g^\tau \cdot \theta'_\Omega\, d\mathcal{H}_2, \quad \forall \theta' \in G^{\rm I_{T2}}. \tag{4.5}$$

Besides the singular case $\mathrm{I_{M2}} = 0$, we may distinguish two main cases depending on whether e_B^v depends explicitly on v_B ($\mathrm{I_{M2}} = 1, 2$) or not ($\mathrm{I_{M2}} = 3, 4$).

When $\mathrm{I_{M2}} = 1, 2$, as $e^v_B = \partial_3 v_B \otimes_S e_3$, one has:

$$\begin{cases} \mathrm{I_{M2}} = 1, \quad V^{\mathrm{I_{M2}}} := \Big\{(v_\Omega, v_B) \in H^1_{\Gamma_{\mathcal{M}D}}(\Omega \setminus S; \mathbb{R}^3) \times H_{\partial_3}(B; \mathbb{R}^3) \text{ s.t. } \gamma_S(v^\pm_\Omega) = \gamma_{S^\pm}(v_B)\Big\} \\ \mathrm{I_{M2}} = 2, \quad V^{\mathrm{I_{M2}}} := \Big\{(v_\Omega, v_B) \in H^1_{\Gamma_{\mathcal{M}D}}(\Omega; \mathbb{R}^3) \times H_{\partial_3}(B; \mathbb{R}^3) \text{ s.t. } \partial_3 v_B = 0, \gamma_S(v_\Omega) = \gamma_{S^\pm}(v_B)\Big\} \end{cases} \tag{4.6}$$

$$\mathcal{M}_\Omega(v'_\Omega) := \int_{\Omega\setminus S} \rho \ddot{u}_\Omega \cdot v'_\Omega + a\big(e(u_\Omega) - \theta_\Omega \alpha I\big) \cdot e(v'_\Omega)\, dx \tag{4.7}$$

$$\mathcal{M}_B(v'_B) := \int_B \bar{\rho}_L \ddot{u}_B \cdot v'_B + \bar{\mu}^{\mathrm{I}}_L a_L \big(\partial_3 u_B \otimes_S e_3 - \bar{\alpha}^I_L \theta_B I\big) \cdot \partial_3 v'_B \otimes_S e_3\, dx. \tag{4.8}$$

When $\mathrm{I_{M2}} = 2$, $\mathcal{M}_B(v'_B)$ is equal to $2 \int_S \bar{\rho}_L \ddot{u}_\Omega \cdot v'_\Omega\, d\mathcal{H}_2$ and vanishes when $\bar{\rho}_L = 0$. One deduces:

$$-[\![\sigma e_3]\!] = 2\bar{\rho}_L \gamma_S(\ddot{u}_\Omega) \tag{4.9}$$

so that the two adhering bodies are stuck together (*i.e.* $[\![u_\Omega]\!] = 0$) when $\bar{\rho}_L > 0$ and perfectly stuck together (*i.e.* $[\![u_\Omega]\!] = [\![\sigma_\Omega e_3]\!] = 0$) when $\bar{\rho}_L = 0$.

When $\mathrm{I_{M2}} = 1$, one has:

$$\mathcal{M}_B(v'_B) = \int_B \bar{\rho}_L \ddot{u}_B \cdot v'_B + \bar{\mu}^{\mathrm{I}}_L a_L \big(\partial_3 u_B \otimes_S e_3 - \bar{\alpha}^{\mathrm{I}}_L \theta_B I\big) \cdot \partial_3 v'_B \otimes_S e_3\, dx \tag{4.10}$$

and the mechanical contact condition reads as:

$$\mp \sigma^\pm_\Omega e_3 = \frac{1}{2}\int_{-1}^{1} \bar{\rho}_L (1 + x_3) \ddot{u}_B\, dx_3 + \bar{\mu}^I_L a_L \left([\![u_\Omega]\!] \otimes_S e_3 - \bar{\alpha}^{\mathrm{I}}_L \Big(\frac{1}{2}\int_{-1}^{1} \theta_B\, dx_3\Big) I \right) e_3. \tag{4.11}$$

Therefore, if $\bar{\rho}_L = 0$ and θ_B is *independent of* x_3 (*i.e.* $\mathrm{I_{T2}} > 1$), one deduces:

$$u_B = \mathrm{Aff}(u_\Omega) \tag{4.12}$$

$$\mathrm{Aff}(u_\Omega) := \langle u_\Omega \rangle + \frac{1}{2} x_3 [\![u_\Omega]\!], \quad \langle u_\Omega \rangle := \frac{1}{2}\Big(\gamma_S(u^+_\Omega) + \gamma_S(u^-_\Omega)\Big) \tag{4.13}$$

$$\mathcal{M}_B(v'_B) = 2\bar{\mu}^{\mathrm{I}}_L \int_S a_L \Big([\![u_\Omega]\!] \otimes_S e_3 - \bar{\alpha}^{\mathrm{I}}_L \gamma_S(\theta_\Omega) I\Big) \cdot [\![v'_\Omega]\!] \otimes_S e_3\, d\widehat{x} \tag{4.14}$$

$$\mp \sigma^\pm_\Omega e_3 = \bar{\mu}^{\mathrm{I}}_L a_L \Big([\![u_\Omega]\!] \otimes_S e_3 - \bar{\alpha}^{\mathrm{I}}_L \gamma_S(\theta_\Omega) I\Big) e_3. \tag{4.15}$$

There is an elastic pull-back with a residual term between the two adhering bodies. The general case for the limit mechanical behavior of the adhesive layer, which clearly appears as a continuous distribution of thermoelastic strings orthogonal to S, will be discussed further.

As regards to the thermal behavior, one has:

$$\mathcal{T}_\Omega(\theta'_\Omega) = \int_{\Omega\setminus S} \beta \theta_\Omega \theta'_\Omega + \kappa \nabla\theta_\Omega \cdot \nabla\theta'_\Omega + \big(a e(\dot{u}_\Omega) \cdot \alpha I\big)\theta'_\Omega\, dx \tag{4.16}$$

$$\mathcal{T}_B(\theta'_B) = \int_B \bar{\beta}_L \theta_B \theta'_B + \bar{\kappa}^{\mathrm{I}}_L g^\theta_B \cdot g^{\theta'}_B + \bar{\mu}^{\mathrm{I}}_L \bar{\alpha}^{\mathrm{I}}_L \big(a_L(\partial_3 \dot{u}_B \otimes_S e_3) \cdot I\big)\theta'_B\, dx. \tag{4.17}$$

We may therefore distinguish two main cases: $\mathrm{I_{T2}} \leq 2$ when $g^\theta_B = 0$ or $(0, \partial_3\theta_B)$ and $\mathrm{I_{T2}} = 3, 4$ when $g^\theta_B = (\widehat{\nabla}\theta_B, 0)$ or 0.

If $I_{T2} \leq 2$, the limit thermal behavior of the adhesive could be considered as the one of a continuous distribution of strings orthogonal to S with specific heat coefficient $\bar{\beta}_L$, thermal conductivity $\bar{\kappa}^I_L$ (thus insulating strings when $I_{T2} = 0$ whereas perfectly conducting ones when $I_{T2} = 0$) subjected to heat sources of lineic density $\bar{\mu}^I_L \bar{\alpha}^I_L a_L\big(\partial_3 \dot{u}_B \otimes_S e_3\big) \cdot I$ (vanishing when $I_{M2} = 0$ or 2). Therefore the thermal contact condition between the adherent bodies reads as:

- $I_{T2} = 0,\ [\![\theta_\Omega]\!] \neq 0,\ q^\pm_\Omega \cdot e_3 = 0$: perfectly insulating interface,

- $I_{T2} = 2,\ [\![\theta_\Omega]\!] = 0,\ -[\![q_\Omega \cdot e_3]\!] = 2\bar{\beta}_L \gamma_S(\dot{\theta}_\Omega) + \bar{\mu}^I_L \bar{\alpha}^I_L \Big([\![\dot{u}_\Omega]\!] \otimes_S e_3\Big) \cdot a_L I$: imperfect thermal contact, perfect when $I_{T1} = 0$ and $I_{M2} = 0$ or 2.

When $I_{T2} = 1$ we have:

$$\mp (q^\pm_\Omega \cdot e_3) = \frac{1}{2} \int_{-1}^{1} (1 + x_3)\Big(\bar{\beta}_L \dot{\theta}_B + \bar{\mu}^I_L \bar{\alpha}^I_L a_L\big(\partial_3 \dot{u}_B \otimes_S e_3\big) \cdot I\Big)\, dx_3 + \bar{\kappa}^I_L [\![\theta_\Omega]\!] \tag{4.18}$$

thus, when $\bar{\beta}_L = 0$, there exists a contact conduction whose contact conductance is $\bar{\kappa}^I_L$ and a source $\bar{\mu}^I_L \bar{\alpha}^I_L a_L [\![\dot{u}_\Omega]\!] \otimes_S e_3 \cdot I$. The general case will be treated further.

When $I_{T2} = 3, 4$, we are indeed dealing with a material surface with a specific heat coefficient $\bar{\beta}_L$, a thermal conductivity $\bar{\kappa}^I_L$ (thus perfectly conducting when $I_{T2} = 4$) subjected to a heat source $\bar{\mu}^I_L \bar{\alpha}^I_L a_L [\![\dot{u}_\Omega]\!] \otimes_S e_3 \cdot I$. So, the imperfect thermal contact condition reads as (the symbol $\widehat{\Delta}$ denoting the Laplacian with respect to the sole coordinates x_1 and x_2):

- $I_{T2} = 3,\ [\![\theta_\Omega]\!] = 0,\ -[\![q_\Omega \cdot e_3]\!] = 2\bar{\beta}_L \gamma_S(\dot{\theta}_\Omega) - \bar{\kappa}^I_L \widehat{\Delta} \gamma_S(\theta_s) + \bar{\mu}^I_L \bar{\alpha}^I_L [\![\dot{u}_\Omega]\!] \otimes_S e_3 \cdot a_L I$,

- $I_{T2} = 4,\ [\![\theta_\Omega]\!] = 0, \gamma_S(\theta_\Omega) = C$, possibly T_0 if $\mathcal{H}_1(\gamma^{\mathcal{T}\mathrm{D}}) > 0, -[\![q_\Omega \cdot e_3]\!] = 2\bar{\beta}_L \gamma_S(\dot{\theta}_s) + \bar{\mu}^I_L \bar{\alpha}^I_L [\![\dot{u}_\Omega]\!] \otimes_S e_3 \cdot a_L I$,

the thermal contact being perfect when $I_{T1} = 0$ and $I_{M2} = 2$.

When $I_{M2} = 0$, $\mathcal{M}_B(v'_B)$ reduces to $\displaystyle\int_B \bar{\rho}_L \ddot{u}_B \cdot v'_B\, dx$ which implies $\sigma^I_\Omega e_3 = 0$: the two adhering bodies are free to separate. The thermal contact condition was detailed in the previous study of the case $I_{M2} = 1$.

When $I_{M2} = 3, 4$, by using $e^\Psi_B = \Psi \otimes_S e_3$, Ψ arbitrary in $C^\infty_0(B; \mathbb{R}^3)$, (4.1) yields $\big[a_L\big(e^u_B - \alpha^I_L \bar{\theta}_B I\big)\big]^\perp = 0$. Then, if $\widetilde{a}_L$ is the element of $\mathrm{Lin}(\mathbb{S}^2)$ defined by:

$$\widetilde{a}_L q \cdot q := \mathrm{Inf}\, \Big\{a_L e \cdot e\, ;\ e \in \mathbb{S}^3 \text{ s.t. } \widehat{e} = q\Big\},\ \forall q \in \mathbb{S}^2 \tag{4.19}$$

one has $\widehat{a_L\big(e^u_B - \bar{\alpha}^I_L \theta_B I\big)} = \widetilde{a}_L\Big(\widehat{e(u_B)} - \bar{\alpha}^I_L \theta_B \widehat{I}\Big)$ and consequently:

$$\mathcal{M}_B(v'_B) = \int_B \bar{\rho}_L \ddot{\mathring{u}}_B \cdot \mathring{v}'_B + \bar{\mu}^I_L \widetilde{a}_L \Big(\widehat{e(u_B)} - \bar{\alpha}^I_L \theta_B \widehat{I}\Big) \cdot \widehat{e(v'_B)}\, dx,\ \forall (v'_\Omega, v'_B) \in V^{I_{M2}} \tag{4.20}$$

$$\mathcal{T}_B(\theta'_B) = \int_B \Big(\bar{\beta}_L + \bar{\mu}^I_L (\bar{\alpha}^I_L)^2 \Big(a_L I \cdot I - \widetilde{a}_L \widehat{I} \cdot \widehat{I}\Big)\Big) \dot{\theta}_B \theta'_B + \bar{\kappa}^I_L g^\theta_B \cdot g^{\theta'}_B + \bar{\mu}^I_L \bar{\alpha}^I_L \Big(\widetilde{a}_L \widehat{e}(\dot{u}_B) \cdot \widehat{I}\Big) \theta'_B\, dx,\ \forall (\theta'_\Omega, \theta'_B) \in G^{I_{M2}}_\theta \tag{4.21}$$

$$V^3 := \Big\{v = (v_\Omega, v_B) \in H^1_{\Gamma_{\mathcal{M}\mathrm{D}}}(\Omega; \mathbb{R}^3) \times V_{\mathrm{KL}}(B); \gamma_S(u_{\Omega 3}) = C,\ u^M_B = \gamma_S(\widehat{u}_\Omega),\ \widehat{\nabla} u^F_B = 0\Big\} \tag{4.22}$$

$$V^4 := \Big\{v = (v_\Omega, v_B) \in H^1_{\Gamma_{\mathcal{M}\mathrm{D}}}(\Omega; \mathbb{R}^3) \times V_{\mathrm{KL}}(B); \widehat{e}\big(\gamma_S(\widehat{u}_\Omega)\big) = 0,\ u^F_B = \gamma_S(u_{\Omega 3})\Big\}. \tag{4.23}$$

An interesting phenomena is highlighted: *the appearance of an added specific heat coefficient*

$$\bar{\beta}_L^{\mathrm{add}} := \bar{\mu}_L^{\mathrm{I}}\big(\bar{\alpha}_L^{\mathrm{I}}\big)^2\Big(a_L I \cdot I - \widetilde{a}\,\widehat{I} \cdot \widehat{I}\Big) \tag{4.24}$$

always positive unless $\bar{\alpha}_L^{\mathrm{I}}(aI)^{\perp} = 0$, in the limit behavior of the adhesive layer, while the stiffness involves $\widetilde{a}$ in place of a as in the Kirchhoff-Love anisotropic plates theory (see [10, 16, 17]).

So, when $\mathrm{I_{M2}} = 3$, we have

$$\mathcal{M}_B(v'_B) = 2\int_S \bar{\rho}_L \gamma_S(\widehat{\ddot{u}}_\Omega)\cdot\gamma_S(\widehat{v}'_\Omega) + \bar{\mu}_L^{\mathrm{I}}\widetilde{a}_L\Big(\widehat{e}\big(\gamma_S(\widehat{u}_\Omega)\big) - \frac{1}{2}\bar{\alpha}_L^{\mathrm{I}}\int_{-1}^{1}\theta_B\,dx_3\widehat{I}\Big)\cdot\widehat{e}\big(\gamma_S(\widehat{v}'_\Omega)\big)\,dx \tag{4.25}$$

and the mechanical contact condition between the adhering bodies reads as:

- $[\![u_\Omega]\!] = 0$
- $u_{\Omega 3}$ constant on S, $\displaystyle\int_S(\sigma_\Omega^{\pm}e_3)\cdot e_3\,d\mathcal{H}_2 = 0$
- $-[\![\widehat{\sigma_\Omega e_3}]\!] = 2\left(\bar{\rho}_L\gamma_S(\widehat{\ddot{u}}_\Omega) - \bar{\mu}_L^{\mathrm{I}}\widehat{\mathrm{div}}\left(\widetilde{a}_L\big(\widehat{e}\big(\gamma_S(\widehat{u}_\Omega)\big)\big) - \frac{1}{2}\bar{\alpha}_L^{\mathrm{I}}\int_{-1}^{1}\theta_B\,dx_3\,\widehat{I}\right)\right).$

It looks like a deformable material surface which is stuck between the adhering bodies and enjoys only in-plane strains.

The nature of the thermal behavior of the adhesive was already examined but here we can make more explicit the thermal contact conditions:

- $\mathrm{I_{T2}} = 0, [\![\theta_\Omega]\!] \neq 0,\ \mp\big(q_\Omega^{\pm}\cdot e_3\big) = 0$: perfectly insulating interface,
- $\mathrm{I_{T2}} = 1, [\![\theta_\Omega]\!] \neq 0,\ \mp(q_\Omega^{\pm}\cdot e_3) = \frac{1}{2}(\bar{\beta}_L+\bar{\beta}_L^{\mathrm{add}})\displaystyle\int_{-1}^{1}(1+x_3)\dot{\theta}_B\,dx_3+\bar{\mu}_L^{\mathrm{I}}\bar{\alpha}_L^{\mathrm{I}}\widetilde{a}_L\widehat{e}\big(\gamma_S(\widehat{\dot{u}}_\Omega)\big)\cdot\widehat{I}+\bar{\kappa}_L^{\mathrm{I}}[\![\theta_\Omega]\!]$: contact conduction,
- $\mathrm{I_{T2}} = 2, [\![\theta_\Omega]\!] = 0, -[\![q_\Omega\cdot e_3]\!] = \big(\bar{\beta}_L + \bar{\beta}_L^{\mathrm{add}}\big)\gamma_S(\dot{\theta}_\Omega) + \bar{\mu}_L^{\mathrm{I}}\bar{\alpha}_L^{\mathrm{I}}\widetilde{a}\widehat{e}(\gamma_S(\widehat{\dot{u}}_\Omega))\cdot\widehat{I}$: imperfect thermal contact,
- $\mathrm{I_{T2}} = 3, [\![\theta_\Omega]\!] = 0, -[\![q_\Omega\cdot e_3]\!] = \big(\bar{\beta}_L+\bar{\beta}_L^{\mathrm{add}}\big)\gamma_S(\dot{\theta}_\Omega)-\bar{\kappa}_L^{\mathrm{I}}\widehat{\Delta}\gamma_S(\theta_\Omega)+\bar{\mu}_L^{\mathrm{I}}\bar{\alpha}_L^{\mathrm{I}}\widetilde{a}_L\widehat{e}\big(\gamma_S(\dot{u}_\Omega)\big)\cdot\widehat{I}$: imperfect thermal contact,
- $\mathrm{I_{T2}} = 4, [\![\theta_\Omega]\!] = 0, \gamma_S(\theta_\Omega)$ constant on S, possibly equal to T_0 if $\mathcal{H}_1(\gamma^{\mathcal{T}\mathrm{D}}) > 0$, $-[\![q_\Omega\cdot e_3]\!] = \big(\bar{\beta}_L+\bar{\beta}_L^{\mathrm{add}}\big)\gamma_S(\dot{\theta}_\Omega)+\bar{\mu}_L^{\mathrm{I}}\bar{\alpha}_L^{\mathrm{I}}\widehat{e}\big(\gamma_S(\dot{u}_\Omega)\big)\cdot\widehat{I}$: imperfect thermal contact.

When $\mathrm{I_{M2}} = 4$ one has:

$$\begin{aligned}\mathcal{M}_B(v'_B) = 2\int_S \bar{\rho}_L\ddot{u}_{\Omega 3}v'_{\Omega 3} + \bar{\mu}_L^{\mathrm{I}}\Big(\widetilde{a}_L\widehat{e}(u_B^M)\cdot\widehat{e}(v_B'^M) + \frac{1}{3}\widetilde{a}_L\widehat{D^2}\gamma_S(u_{\Omega 3})\cdot\widehat{D^2}\gamma_S(v'_{\Omega 3})\\ - \bar{\alpha}_L^{\mathrm{I}}\Big(\frac{1}{2}\int_{-1}^{1}\theta_B\,dx_3\Big)\widetilde{a}_L\widehat{I}\cdot\widehat{e}(v_B'^M) + \int_{-1}^{1}x_3\theta_B\,dx_3\widetilde{a}_L\widehat{I}\cdot\widehat{D^2}\gamma_S v'_{\Omega 3}\Big)\,d\mathcal{H}_2\end{aligned} \tag{4.26}$$

$$\mathcal{T}_B(\theta'_B) = \int_B\big(\bar{\beta}_L + \bar{\beta}_L^{\mathrm{add}}\big)\dot{\theta}_B\theta'_B + \bar{\kappa}_L^{\mathrm{I}}g_B^{\theta}\cdot g_B^{\theta'} + \bar{\mu}_L^{\mathrm{I}}\bar{\alpha}_L^{\mathrm{I}}\Big(\widetilde{a}\big(\widehat{e}(u_B^M) - x_3\widehat{D^2}\gamma_S(u_{\Omega 3})\big)\cdot\widehat{I}\Big)\theta'_B\,dx. \tag{4.27}$$

Because $g_B^{\mathcal{T}} = \big(\widehat{\nabla}\mathcal{T}_B, 0\big)$ when $\mathrm{I}_{\mathrm{T}2} = 3$, the limit behavior of the adhesive layer is then similar to the one observed by [18] for thin linearly thermoelastic plates: a flexural problem for the component of the displacement field normal to S with a coupled membrane-thermal problem for the in-plane component of the displacement and the temperature.

The mechanical contact condition between the adhering bodies reads as:

- $[\![u_\Omega]\!] = 0$
- $\widehat{e}\big(\gamma_S(\widehat{u}_\Omega)\big) = 0$ (4.28)
- $-[\![(\sigma_\Omega e_3)\cdot e_3]\!] = 2\bar{\rho}_L\gamma_S(\ddot{u}_{\Omega 3}) + \dfrac{1}{3}\widehat{D^2}\big(\widetilde{a}_L\widehat{D^2}\gamma_S(u_{\Omega 3})\big)$

the material surface inserted between the two adhering bodies may be considered as a second-grade elastic one, enjoying only a motion orthogonal to S (a flexural problem ...).

On the other hand, the thermal contact condition reads as:

$$[\![\theta_\Omega]\!] = 0, \quad -[\![q_\Omega\cdot e_3]\!] = 2\Big[\big(\bar{\beta}_L + \bar{\beta}_L^{\mathrm{add}}\big)\dot{\theta}_\Omega - \bar{\kappa}_L^{\mathrm{I}}\widehat{\Delta}\gamma_S(\theta_\Omega) + \bar{\mu}_L^{\mathrm{I}}\bar{\alpha}_L^{\mathrm{I}}\widetilde{a}_L\widehat{e}(u_B^M)\cdot\widehat{I}\Big] \tag{4.29}$$

it involves the additional variable u_B^M defined on S with values in $\mathbb{R}^2$ satisfying

$$\widehat{\sigma}_B = \bar{\mu}_L^{\mathrm{I}}\widetilde{a}_L\Big(\widehat{e}(u_B^M) - \bar{\alpha}_L^{\mathrm{I}}\gamma_S(\theta_\Omega)\widehat{I}\Big), \quad \widehat{\mathrm{div}}\widehat{\sigma}_B = 0. \tag{4.30}$$

All this corresponds to a thermomechanical material surface occupying S whose material constants are given by $\bar{\beta}+\bar{\beta}_L^{\mathrm{add}}$, $\bar{\kappa}_L^{\mathrm{I}}$, $\bar{\alpha}_L^{I}$, $\bar{\mu}_L^{\mathrm{I}}\widetilde{a}$ subjected to an inner heat source and free of mechanical loading. Of course u_B^M may be eliminated and consequently the thermal contact condition along S is a non local relation (in time, only) between the normal flux $(q_\Omega^{\pm}\cdot e_3)(\widehat{x}, t)$ at the courant time t and the whole history of $\gamma_S(\theta_\Omega)(\widehat{x}, \tau)$, $0 \le \tau \le t$.

When $\mathrm{I}_{\mathrm{T}2} = 4$, $\bar{\kappa}_L^{\mathrm{I}} = \infty$ compels $\widehat{\nabla}\gamma_S(\theta_\Omega) = 0$, so that S is an isothermal surface possibly at T_0 when $\mathcal{H}_1(\gamma^{\mathcal{T}\mathrm{D}}) > 0$, while the mechanical contact is similar to (4.28), (4.30).

This is also the case when $\mathrm{I}_{\mathrm{T}2} = 0$ or 2 but with thermal conditions like:

- $\mathrm{I}_{\mathrm{T}2} = 0, \quad [\![\theta_\Omega]\!] \neq 0, \quad \mp q_\Omega^{\pm}\cdot e_3 = 0:$ perfectly insulating wall,
- $\mathrm{I}_{\mathrm{T}2} = 2, \quad [\![\theta_\Omega]\!] = 0, \quad -[\![q_\Omega\cdot e_3]\!] = 2\big(\bar{\beta}_L + \bar{\beta}_L^{\mathrm{add}}\big)\gamma_S(\dot{\theta}_\Omega) + \bar{\mu}_L^{\mathrm{I}}\bar{\alpha}_L^{\mathrm{I}}\widetilde{a}_L\widehat{e}(u_B^M)\cdot\widehat{I}.$

When $\mathrm{I}_{\mathrm{T}1} = 1$, as the additional temperature variable θ_B does depend on x_3, the limit thermoelastic behavior of the adhesive layer cannot be interpreted in terms of a material surface.

Thus, in order to clarify the thermomechanical condition between the two adhering bodies, difficulties occur when $\mathrm{I}_{\mathrm{M}2} = 1$ or 4 and/or $\mathrm{I}_{\mathrm{T}2} = 1$, which correspond to the cases when the additional state variable u_B depends on x_3 ($\mathrm{I}_{\mathrm{M}2} = 1$), does not explicitly depends on the traces on S of the displacements of the adhering bodies ($\mathrm{I}_{\mathrm{M}2} = 4$) and/or the additional state variable θ_B depends on x_3 ($\mathrm{I}_{\mathrm{T}2} = 1$). In some of these cases, by adding a condition like $\mathrm{I}_{\mathrm{M}1} = 0$ (light adhesive layer) or $\mathrm{I}_{\mathrm{T}1} = 0$ (low specific heat coefficient) we again meet thermomechanical contact conditions involving the traces on S of the state variables of the adhering bodies only as in the cases we listed previously.

When $\mathrm{I}_{\mathrm{M}2} = 1$, $u_B^0 := u_B - \mathrm{Aff}(u_\Omega)$ satisfies

$$\begin{cases} u_B^0 \in H_{\partial_3, S^+\cup S^-}(B;\mathbb{R}^3) := \big\{u \in H_{\partial_3}(B;\mathbb{R}^3) \text{ s.t. } \gamma_{S^\pm}(u) = 0\big\}, \\ \displaystyle\int_B \bar{\rho}_L\ddot{u}_B^0\cdot v' + \bar{\mu}_L^{\mathrm{I}}a_L\big(\partial_3 u_B^0 \otimes_S e_3\big)\cdot\big(\partial_3 v' \otimes_S e_3\big)\,dx \\ \qquad = -\displaystyle\int_B \rho_L\ddot{\mathrm{Aff}}(u_\Omega)\cdot v' + \theta_B\bar{\mu}_L^{\mathrm{I}}\bar{\alpha}_L^{\mathrm{I}}a_L\partial_3 v'_B \otimes_S e_3\cdot I\,dx, \ \forall v' \in H_{\partial_3, S^+\cup S^-}(B;\mathbb{R}^3) \end{cases} \tag{4.31}$$

so that except when $I_{M1} = 0$ and $I_{T2} \neq 0, 1$, u_B differs from $\mathrm{Aff}(u_\Omega)$ and the mechanical condition (4.11) does not involves the instantaneous values of the traces on S of $u^\pm_\Omega$ and $\theta^\pm_\Omega$. Of course, as the equations governing the evolutions of u_B and θ_B can be solved in terms of the whole history of the traces on S of $u^\pm_\Omega$ and $\theta^\pm_\Omega$, the contact condition at $(\widehat{x}, t)$ is a rather complex function of the history of $\gamma_S(u^\pm_\Omega)(\widehat{x}, \cdot)$ and $\gamma_S(\theta^\pm_\Omega)(\widehat{x}, \cdot)$ and not only of the history of the jumps $[\![u_\Omega]\!](\widehat{x}, \cdot)$, $[\![\theta_\Omega]\!](\widehat{x}, \cdot)$.

When $I_{T2} = 1$, if $\mathrm{Aff}(\theta_\Omega)$ is defined similarly as $\mathrm{Aff}(u_\Omega)$ (see (4.13)), $\theta^0_B := \theta_B - \mathrm{Aff}(\theta_\Omega)$ satisfies:

$$\begin{cases} \theta^0_B \in H_{\partial_3, S^+\cup S^-}(B), \\ \displaystyle\int_B \bar{\beta}_L \dot{\theta}^0_B \theta' + \bar{\kappa}^{\mathrm{I}}_L \partial_3\theta^0_B \partial_3\theta'_B \, dx = -\int_B \Big(\bar{\beta}_L \dot{\mathrm{Aff}}(\theta_\Omega) + \bar{\mu}^{\mathrm{I}}_L \bar{\alpha}^{\mathrm{I}}_L a_L\big(\partial_3 \dot{u}_B \otimes_S e_3\big) \cdot I\Big)\theta' \, dx, \quad \text{if } I_{M2} = 1,2 \\ \displaystyle\int_B \big(\bar{\beta}_L + \bar{\beta}^{\mathrm{add}}_L\big)\dot{\theta}^0_B \theta'_B + \bar{\kappa}^{\mathrm{I}}_L \partial_3\theta^0_B \partial_3\theta'_B \, dx = \begin{cases} \displaystyle -\int_B \Big(\big(\bar{\beta}_L + \bar{\beta}^{\mathrm{add}}_L\big)\dot{\mathrm{Aff}}(\theta_\Omega) + \bar{\mu}^{\mathrm{I}}_L \bar{\alpha}^{\mathrm{I}}_L \widetilde{a}_L \widehat{e}\big(\gamma_S(\widehat{u}_\Omega)\big) \cdot \widehat{I}\Big)\theta' \, dx, \ \text{if } I_{M2} = 3 \\ \hfill \forall \theta' \in H_{\partial_3, S^+\cup S^-}(B) \\ \displaystyle -\int_B \Big(\big(\bar{\beta}_L + \bar{\beta}^{\mathrm{add}}_L\big)\dot{\mathrm{Aff}}(\theta_\Omega) - x_3 \bar{\mu}^{\mathrm{I}}_L \bar{\alpha}^{\mathrm{I}}_L \widetilde{a}_L \widehat{D^2}\gamma_S(u_{\Omega 3}) \cdot \widehat{I}\Big)\theta' \, dx, \ \text{if } I_{M2} = 4. \end{cases} \end{cases} \tag{4.32}$$

Except when $(I_{M2}, I_{T1}) = (2, 0)$, $\mathrm{Aff}(\theta_\Omega)$ does not solve (4.32), the thermal contact condition which at time t reads as:

$$\mp q^\pm_\Omega \cdot e_3 = \begin{cases} \displaystyle \frac{1}{2}\int_{-1}^{1} (1 + x_3)\bar{\beta}_L \dot{\theta}_B + \bar{\mu}^{\mathrm{I}}_L \bar{\alpha}^{\mathrm{I}}_L a_L\big(\partial_3 \dot{u}_B \otimes_S e_3\big) \cdot I \, dx_3 + \bar{\kappa}^{\mathrm{I}}_L [\![\theta_\Omega]\!], \quad \text{if } I_{M2} = 1,2 \\ \displaystyle \frac{1}{2}\int_{-1}^{1} (1 + x_3)\big(\bar{\beta}_L + \bar{\beta}^{\mathrm{add}}_L\big)\dot{\theta}_B \, dx_3 + \bar{\mu}^{\mathrm{I}}_L \bar{\alpha}^{\mathrm{I}}_L \widetilde{a}_L \widehat{e}\big(\gamma_S(\widehat{u}_\Omega)\big) \cdot \widehat{I} + \bar{\kappa}^{\mathrm{I}}_L [\![\theta_\Omega]\!], \quad \text{if } I_{M2} = 2,3 \\ \displaystyle \frac{1}{2}\int_{-1}^{1} (1 + x_3)\big(\bar{\beta}_L + \bar{\beta}^{\mathrm{add}}_L\big)\dot{\theta}_B + \bar{\mu}^{\mathrm{I}}_L \bar{\alpha}^{\mathrm{I}}_L \widetilde{a}_L \widehat{D^2}\gamma_S(u_{\Omega 3}) \cdot \widehat{I} x_3 \, dx_3 + \bar{\kappa}^{\mathrm{I}}_L [\![\theta_\Omega]\!], \quad \text{if } I_{M2} = 4 \end{cases} \tag{4.33}$$

involves the whole history of $\gamma_S(u^\pm_\Omega)$ and $\gamma_S(\theta^\pm_\Omega)$.

So, in every cases, the limit thermomechanical behavior of the two adhering bodies and of the adhesive layer are of *the same (thermoelastic) type as that of the original situation*. But, of course, peculiarities of the limit behavior of the layer and the thermomechanical contact condition which replaces it strongly depend on the relative behaviors of the geometric and thermomechanical parameters. *The thermomechanical coupling perpetuates* when $\bar{\mu}^{\mathrm{I}}_L \bar{\alpha}^{\mathrm{I}}_L e^u_B$ does not vanish which is the case when I_{M2} differs from 0 or 2 with $\bar{\alpha}^{\mathrm{I}}_L$ positive.

5. Concluding remarks

This rather lengthy and complex thermomechanical presentation of the results of our mathematical analysis exemplifies the flexibility of use but also the power of Trotter's theory of approximation of semi-groups of operators acting on variable spaces: it permits a unitary treatment with very few technicalities.

Our proposal of simplified but accurate enough models for the behavior of the structure made of the two adhering bodies and the thin adhesive layer, which has to be formulated on the genuine reference configurations $\Omega^\pm_\varepsilon$ and B_ε is of course obtained through the Trotter representant $P^{\mathrm{I}}_s U^{\mathrm{I}}$ of the solution U^{I} of the limit problem (3.42), *s taking the values of the original data*. When I_{M2} differs from 0, a variant of $P^{\mathrm{I}}_s U^{\mathrm{I}}$ may be used through the construct detailed in the proof of Proposition 3.3. Thus, from a computational and practical point of view, a finite element approximation can be implemented without meshing the thin layer occupied by the adhesive !

It should be noted that, contrary to the cumbersome method - frequent in the literature - consisting of firstly switching back to a fixed abstract domain through a "scaling" (change of coordinates and unknowns), abstract domain where the convergence is formally or rigorously studied and secondly returning - but not always - to the initial physical domain, we have hereby treated *directly* through the representation operator P^{I}_s the convergence of the initial problem where, obviously, the limit can be, according to index I, expressed in a fixed abstract domain defined through the

”scaling” outlined above *but* which is used only when it is necessary to *refine* the determination of the asymptotic behavior of sequences of thermomechanical states with bounded energies.

Eventually the present study which corrects and improves [6] may be considered as a framework to assess the formal and partial modeling proposed in [19] concerning poroelasticity as it is well-known that equations involved in linear poroelasticity are the same as those in linear thermoelasticity.

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